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Algebra and Trigonometry

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Preface

The authors believe that in this book the basic material of college algebra and trigonometry has been presented with sufficient rigor to provide a firm and coherent groundwork for subsequent courses in mathematics. The material is presented in such a way that it can be grasped by the student without undue assistance.

The first chapter consists of a number of introductory topics which are intended to serve as a review of elementary algebra. Actually, something more than a mere review is available in this chapter. Not only are the review topics considered from a more mature point of view than is usual, but the treatment is interwoven with concepts that are basic for an understanding of more advanced mathematical topics. We begin with the algebra of the real-number system. Axioms pertaining to fundamental operations are given, and the various rules for the elementary operations of algebra are derived and logically connected with the basic assumptions. We are led naturally to an ordering of the real-number system and to the foundation for a later chapter on inequalities that is easier to understand and more useful than the treatment one customarily finds in textbooks.

The second chapter introduces the student to the function concept, which serves as a basis for much of the remaining work of the book. Certain aspects of the discussion become somewhat abstract, but the student is reminded that a proper understanding of the true nature of a function is important for virtually all later courses in mathematics.

In line with modern demands, the trigonometric functions are initially introduced in the third chapter as functions of real numbers. Following this presentation, the transition to functions of angles is relatively simple.

The rest of the volume contains all the usual topics from college algebra and trigonometry. In certain instances a particular development may differ somewhat from that usually found. In such cases, the authors believe, the departure is to the advantage of the student.

We are indebted to our colleague, Professor Morris Dansky, for his valuable suggestions while the manuscript was in preparation. We wish particularly to express our deep appreciation to Professor L. R. Wilcox for his thorough criticism of the manuscript and his invaluable suggestions for improvement of the text. Finally, a special word of thanks is due the International Textbook Company for its cooperation and patience.

A. K. BETTINGER
J. A. ENGLUND

Omaha, Nebraska
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1 Introductory Topics

1–1. THE REAL-NUMBER SYSTEM

The real-number system that we use in the early part of this course is a development from the original counting numbers, or *positive integers*, such as 1, 2, and 3. Almost simultaneously with the invention of positive integers, practical problems of measurement gave rise to positive fractions, such as 1/2, 5/6, and 16/7. Much later, in comparatively modern times, the concepts of negative numbers and of other types of numbers were gradually developed. Negative numbers were invented when the problem of subtracting one number from a smaller one presented itself. Thus, the number system was soon enlarged to include the negative integers and fractions. These positive and negative numbers, together with zero, are called the *rational numbers*. Hence, a rational number is defined to be any number that can be expressed as the quotient, or ratio, of two integers. For example, $-2/3$, 5 (which may be considered as $5/1$), and -7 are rational numbers.

The number system was then extended to include also numbers which cannot be expressed as the quotient of two integers, namely, the *irrational numbers;* examples are $\sqrt{2}$ and π. The two classes of numbers, rational and irrational, comprise the *real numbers.* These numbers are so called in contrast to the *imaginary* or *complex* numbers considered in Chapter 11.

1–2. FUNDAMENTAL ASSUMPTIONS

We shall proceed to introduce the four fundamental operations of addition, subtraction, multiplication, and division into the system of real numbers. The reader has probably been performing these operations in arithmetic and algebra without being conscious that certain basic laws were being obeyed. We shall introduce the four fundamental operations and state, without proof, the laws or assumptions governing them.

Addition. It is assumed that there is a mode of combining any two real numbers a and b so as to produce a definite real number called their *sum.* This mode of combination is called *addition.* The sum of a and b is denoted by $a + b$. In this sum a and b are called *terms.*

Multiplication. It is assumed that there is a mode of combining any two real numbers a and b to produce a definite real number called their *product.* This mode of combination is called *multiplication.* The product of a and b is denoted by $a \cdot b$ or by ab. The individual numbers a and b are called *factors* of the product.

Commutative Law for Addition. If a and b are any real numbers, then

$$a + b = b + a. \tag{1-1}$$

Thus[1], the sum of two numbers is the same regardless of the order in which they are added. For example,

$$2 + 3 = 3 + 2.$$

Associative Law for Addition. If a, b, c are any real numbers, then

$$(a + b) + c = a + (b + c). \tag{1-2}$$

That is, we obtain the same result whether we add the sum of a and b to c, or we add a to the sum of b and c. Since the way in which we associate or group these numbers is immaterial, we may write this common value as $a + b + c$ without fear of ambiguity. For example,

$$2 + 3 + 4 = (2 + 3) + 4 = 2 + (3 + 4).$$

Commutative Law for Multiplication. If a and b are any real numbers, then

$$ab = ba. \tag{1-3}$$

That is, the product of two numbers is the same regardless of the order in which they are multiplied. For example,

$$2 \cdot 3 = 3 \cdot 2.$$

Associative Law for Multiplication. If a, b, c are any real numbers, then

$$(ab)c = a(bc). \tag{1-4}$$

[1] Illustrations of the laws are given here only for the most familiar numbers, the positive integers. It is understood, however, that the laws apply to all real numbers.

That is, we obtain the same result whether we multiply the product of a and b by c, or we multiply a by the product of b and c. Since the way in which we associate or group these numbers is immaterial, we may write the result as abc without fear of ambiguity. Thus

$$2 \cdot 3 \cdot 4 = (2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4).$$

Distributive Law. If a, b, c are any real numbers, then[2]

$$a(b + c) = ab + ac. \tag{1–5}$$

This law, which is usually known as the distributive law for multiplication with respect to addition, effects a connection between addition and multiplication. The distributive law forms the basis for the factoring process in algebra, as will be seen.

A simple example of the distributive law is

$$2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4.$$

This law can be extended to the case where the sum consists of three or more terms, as in the following illustration:

$$3a(x + 2y - 3z) = 3ax + 6ay - 9az.$$

For positive integers, multiplication may also be interpreted as repeated addition. Thus, by the distributive law,

$$3 \cdot 4 = (1 + 1 + 1) \cdot 4 = (1 \cdot 4) + (1 \cdot 4) + (1 \cdot 4) = 4 + 4 + 4,$$

$$\begin{aligned} 3 \cdot 4 &= 3(1 + 1 + 1 + 1) = (3 \cdot 1) + (3 \cdot 1) + (3 \cdot 1) + (3 \cdot 1) \\ &= 3 + 3 + 3 + 3. \end{aligned}$$

Zero. It is assumed that there is a special number called *zero* and denoted by 0, such that, for every real number a,

$$a + 0 = a. \tag{1–6}$$

For example,

$$3 + 0 = 3, \quad 0 + 1 = 1, \quad 0 + 0 = 0.$$

It can be easily shown that only one number with the property of 0 can exist. For let $0'$ be another such number. Then, since $a + 0 = a$ and $b + 0' = b$ for any numbers a, b, it follows, by taking $a = 0'$ and $b = 0$, that

$$0' + 0 = 0', \text{ and } 0 + 0' = 0.$$

From the commutative law, $0 = 0'$.

[2] The right side of (1–5) should read $(ab) + (ac)$. However, by convention, we agree to omit the parentheses when all multiplications are to be performed before any addition.

Negative of a Number. It is assumed that for every real number a there exists a corresponding number, called the *negative of a* and designated by $-a$, such that

$$a + (-a) = 0. \tag{1-7}$$

For example,

$$1 + (-1) = 0, \quad (-2) + 2 = 0.$$

That each number has but one negative may be shown in the following way: Let x be another negative of a, so that $a + x = 0$. Then

$$-a = (-a) + 0 = (-a) + (a + x).$$

By associativity,

$$-a = ((-a) + a) + x,$$

or

$$-a = 0 + x = x.$$

In particular, the negative of zero is

$$-0 = -0 + 0 = 0.$$

The Unit. It is assumed that there is a special number called the *unit* and denoted by 1, such that, for every real number a,

$$a \cdot 1 = a. \tag{1-8}$$

There cannot be a second unit $1'$. If there were, we could say that

$$1 \cdot 1' = 1, \quad 1' \cdot 1 = 1',$$

whence $1 = 1'$.

Reciprocal of a Number. It is assumed that for every number a which is not 0, there is an associated number $\frac{1}{a}$, called the *reciprocal* of a, such that

$$a \cdot \frac{1}{a} = 1. \tag{1-9}$$

The reader may verify the fact that there is only one reciprocal of each number. Thus, if x is another reciprocal of a, that is, if $a \cdot x = 1$, then $x = \frac{1}{a}$.

It is important to note the restriction $a \neq 0$ in the definition of the reciprocal. In the next section we shall see why this restriction is needed.

Subtraction. The *difference* $a - b$, of any real numbers a and b, is defined by

$$a - b = a + (-b). \tag{1-10}$$

The operation indicated by the sign minus which produces for any two real numbers a and b the real number $a - b$ is called *subtraction.*

Division. The *quotient* a/b or $\frac{a}{b}$ or $a \div b$ of any real numbers a and b, where $b \neq 0$, is defined by

$$\frac{a}{b} = a \cdot \left(\frac{1}{b}\right). \tag{1-11}$$

The operation associating with real numbers a and b ($b \neq 0$) their quotient is called *division.*

It should be noted that subtraction and division are subordinate to addition and multiplication, in that they are defined in terms of these latter. The difference $a - b$ is that number x for which $b + x = a$. Also, the quotient a/b is that number y for which $b \cdot y = a$. It should be noted that $a - a = a + (-a) = 0$ for every number a, and that $a/a = a \cdot (1/a) = 1$ for every number $a \neq 0$.

1–3. OPERATIONS WITH ZERO

It has already been noted that the special number 0 has the property $a + 0 = a$ for every real number a. In particular, we may let $a = 0$ to obtain

$$0 + 0 = 0.$$

It has already been noted that $-0 = 0$, so that $a = a + 0 = a - 0$ for every real number a.

Next, we prove that for every real number a,

$$a \cdot 0 = 0. \tag{1-12}$$

Let $x = a \cdot 0$. Then, by the distributive law,

$$x = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 = x + x.$$

If we add $-x$, we obtain

$$0 = x + (-x) = (x + x) + (-x) = x + (x + (-x)) = x + 0 = x.$$

Since $x = a \cdot 0$, (1–12) is established.

From this last result, it follows that, for $b \neq 0$,

$$\frac{0}{b} = 0 \cdot \left(\frac{1}{b}\right) = 0. \tag{1-13}$$

1–4. RECIPROCALS

It was noted that every non-zero number has a reciprocal. We can now see why 0 cannot have a reciprocal. If 0 has a reciprocal x, then $0 \cdot x = 1$. Since it has been shown that $0 \cdot x = 0$, we would have

to conclude that $0 = 1$. However, if $0 = 1$ is allowed, then for every number a we have

$$a = 1 \cdot a = 0 \cdot a = 0.$$

Hence, 0 would be the only number in the number system. This situation obviously should be ruled out. Therefore, 0 cannot have a reciprocal. Moreover, since $a/b = a \cdot (1/b)$, the quotient a/b is not defined when $b = 0$.

The reciprocal of the product of two non-zero numbers can be expressed in another way:

$$\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}, \tag{1–14}$$

provided that neither a nor b is 0. To prove this result, we begin with

$$\frac{1}{a} \cdot \frac{1}{b} \cdot a \cdot b = \frac{1}{a} \cdot a \cdot \frac{1}{b} \cdot b = 1 \cdot 1 = 1.$$

We then multiply by $\dfrac{1}{a \cdot b}$ to obtain

$$\frac{1}{a \cdot b} = 1 \cdot \frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b} \cdot (a \cdot b) \cdot \frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b} \cdot 1 = \frac{1}{a} \cdot \frac{1}{b} \cdot$$

In the preceding proof, free use has been made of the commutative and associative laws.

Finally, if $a \neq 0$, the reciprocal of the reciprocal of a is a itself. Thus,

$$\frac{1}{1/a} = a. \tag{1–15}$$

Since

$$\frac{1}{1/a} \cdot (1/a) = 1,$$

multiplication by a gives

$$a = 1 \cdot a = \frac{1}{1/a} \cdot (1/a) \cdot a = \frac{1}{1/a} \cdot 1 = \frac{1}{1/a} \cdot$$

1–5. THE REAL-NUMBER SCALE

Real numbers may be represented by points on a straight line. On such a line select an arbitrary point 0 as *origin* and lay off equal unit distances in both directions, as shown in Fig. 1-1. (The

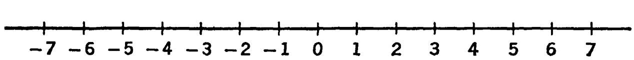

Fig. 1–1

unit segment may have any length whatsoever.) Label the points thus far specified as indicated: 0 is the origin, 1 is the first point to

the right, 2 is the second point to the right, -1 is the first point to the left, and so on. Rational numbers that are not integers correspond to certain other points in a natural way. For example, 1/2 corresponds to the midpoint of the segment joining points labeled 0 and 1; and $-7/3$ represents the point one-third of the distance from the point -2 to the point -3. It is a basic assumption concerning the real numbers that every point corresponds to a unique real number, and that every real number corresponds to exactly one point. The full significance of this assumption cannot be developed in an elementary text.

One observation of importance can be made at this time. The non-zero real numbers are divided into two classes. One class consist of numbers representing points to the "right" of 0, and the other consists of numbers representing points to the "left" of 0. The first class consists of *positive numbers*, and the second of *negative numbers*. The number 0 may be considered as constituting a third class. It is understood that no two of the three classes—zero, positive numbers, and negative numbers—have any numbers in common. Thus a number cannot be both positive and zero, both positive and negative, or both negative and zero. The specific designation of any negative number will include an explicit sign $-$, which is prefixed. (This convention, however, does not exclude the possibility of allowing a general symbol, such as x, to stand for a negative number.) Positive numbers do not require such a sign, although frequently the sign $+$ is used.

It is to be assumed that the sum of two positive numbers is positive, as is also the product of two positive numbers.

1–6. RULES OF SIGNS

To operate effectively with real numbers, a knowledge of the rules of signs and of properties of negative numbers is essential. In each of the following relationships, a and b are any two real numbers, except that the denominator of a fraction may not be zero.

(1–16) $$-(-a) = a.$$

(1–17) $$-(a+b) = -a-b.$$

(1–18) $$-(a-b) = -a+b.$$

(1–19) $$(-a)b = -(ab); \quad \text{in particular, } (-1)b = -b.$$

(1–20) $$(-a)(-b) = ab.$$

(1–21) $$\frac{1}{-b} = -\frac{1}{b}.$$

(1–22) $$\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}.$$

(1–23) $$\frac{-a}{-b} = \frac{a}{b}.$$

Proofs of (1–16) to (1–23):

(1–16) $-(-a) = 0 - (-a) = a + (-a) + (-(-a)) = a + 0 = a.$

(1–17) $$\begin{aligned} -(a+b) &= 0 + 0 - (a+b) = -a + a + (-b) + b - (a+b) \\ &= -a - b + (a+b) - (a+b) \\ &= -a - b + 0 = -a - b \qquad \text{by (1–6), (1–7).} \end{aligned}$$

(1–18) $-(a-b) = -(a + (-b)) = -a - (-b) = -a + b$ by (1–17), (1–16).

(1–19) Since $(-a)\cdot b + a \cdot b = (-a + a)\cdot b = 0 \cdot b = 0$ by (1–12),
$(-a)\cdot b = (-a)\cdot b + a\cdot b - (a\cdot b) = 0 - (a \cdot b) = -(a\cdot b).$

(1–20) $$\begin{aligned} (-a)(-b) &= -((-a)\cdot b) = -(-(a\cdot b)) \\ &= ab \qquad \text{by (1–19), (1–16).} \end{aligned}$$

(1–21) $$\begin{aligned} \frac{1}{-b} &= \frac{1}{-b}\cdot 1 = \frac{1}{-b}\cdot b \cdot \frac{1}{b} = \frac{1}{-b}\cdot(-b)\cdot\left(-\frac{1}{b}\right) \\ &= 1\cdot\left(-\frac{1}{b}\right) = -\frac{1}{b} \qquad \text{by (1–20).} \end{aligned}$$

(1–22) $$\begin{aligned} \frac{a}{-b} &= a \cdot \frac{1}{-b} = a\cdot\left(-\frac{1}{b}\right) = -\left(a\cdot\frac{1}{b}\right) = -\frac{a}{b}; \\ \frac{-a}{b} &= (-a)\cdot\frac{1}{b} = -\left(a\cdot\frac{1}{b}\right) = -\frac{a}{b} \qquad \text{by (1–19).} \end{aligned}$$

(1–23) $$\frac{-a}{-b} = -\left(-\frac{a}{b}\right) = \frac{a}{b} \qquad \text{by (1–22), (1–16).}$$

It has already been observed that non-zero numbers are divided into two classes, namely, *positive* and *negative.* It is assumed that if a is positive, then $-a$ is negative; and that if a is negative, then $-a$ is positive. All calculations involving negative numbers can be made by performing calculations with positive numbers and applying one or more relationships just given. It follows from (1–17), for example, that the sum of two negative numbers is negative, and is equal to the negative of the sum of the negatives of the given numbers. Also, from (1–20) it follows that the product of two negative numbers is positive, and is equal to the product of the negatives of the given numbers. By (1–19) the product of a positive number and a negative number is negative.

1–7. FUNDAMENTAL OPERATIONS ON FRACTIONS

A further study of the algebra of real numbers leads us to the consideration of the fundamental operations as applied to fractions. By definition, a fraction is the quotient obtained by dividing one number a by another number b, where b is not zero. We call a the *numerator* and b the *denominator;* and we generally write the fraction a/b, read "a over b" or "a divided by b."

We shall list the following basic relationships for applying the four operations to fractions. In them a, b, c, d are any real numbers, except that no factor in the denominator of a fraction may be zero.

(1–24) $$\frac{ac}{bc} = \frac{a}{b}.$$

(1–25) $$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

(1–26) $$\frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}.$$

(1–27) $$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

(1–28) $$\frac{a}{c} - \frac{b}{d} = \frac{ad-bc}{cd}.$$

(1–29) $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

(1–30) $$\frac{a}{b} \Big/ \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

A special case of (1–30) is

$$1 \Big/ \frac{c}{d} = \frac{d}{c},$$

which states that the reciprocal of a fraction is found by inverting the fraction. Also, by (1–30), dividing by a fraction is equivalent to multiplying by its reciprocal.

Proofs of (1–24) to (1–30):

(1–29) Since $\frac{1}{b} \cdot \frac{1}{d} = \frac{1}{bd}$ by (1–14),

$$\frac{a}{b} \cdot \frac{c}{d} = a \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d} = (a \cdot c) \cdot \frac{1}{b} \cdot \frac{1}{d} = a \cdot c \cdot \frac{1}{bd} = \frac{ac}{bd}.$$

(1–24) $$\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b} \qquad \text{by (1–29).}$$

(1–25) $$\frac{a}{c} + \frac{b}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c} = (a+b) \cdot \frac{1}{c} = \frac{a+b}{c} \qquad \text{by (1–5).}$$

(1–26) $$\frac{a}{c}+\frac{b}{d}=\frac{a\cdot d}{c\cdot d}+\frac{b\cdot c}{c\cdot d}=\frac{ad+bc}{cd} \quad \text{by (1–24), (1–25).}$$

(1–27) $$\frac{a}{c}-\frac{b}{c}=\frac{a}{c}+\frac{(-b)}{c}=a\cdot\frac{1}{c}+(-b)\cdot\frac{1}{c}=(a-b)\frac{1}{c}$$
$$=\frac{a-b}{c} \quad \text{by (1–22), (1–5).}$$

(1–28) $$\frac{a}{c}-\frac{b}{d}=\frac{a}{c}+\frac{(-b)}{d}=\frac{a\cdot d}{c\cdot d}+\frac{(-b)\cdot c}{c\cdot d}$$
$$=\frac{ad-bc}{cd} \quad \text{by (1–22), (1–24), (1–27).}$$

(1–30) $$\frac{a}{b}\Big/\frac{c}{d}=\frac{a}{b}\cdot\frac{1}{\frac{c}{d}}=\frac{a}{b\cdot\frac{c}{d}}=\frac{ad}{b\cdot\frac{c}{d}\cdot d}=\frac{ad}{b\cdot\frac{cd}{1\cdot d}}=\frac{ad}{bc}$$
$$=\frac{a}{b}\cdot\frac{d}{c} \quad \text{by (1–29), (1–24).}$$

Example 1–1. State which of the fundamental assumptions are employed in each of the following equations:

a) $3+9+(-5)=3+(-5)+9$.
b) $11+(6+3)+7=(11+6)+(3+7)$.
c) $(2\cdot 3)\cdot 5=2\cdot(3\cdot 5)$.
d) $5(3+4)=5\cdot 3+5\cdot 4$.

Solution:

a) The associative and commutative laws for addition.
b) The associative law for addition.
c) The associative law for multiplication.
d) The distributive law.

Example 1–2. In each of the following, perform the indicated operation:

a) $2+3$. *b*) $(-2)+(-3)$. *c*) $5+(-3)$.
d) $(+5)-(+3)$. *e*) $(-7)-(+6)$. *f*) $(-7)-(-6)$.

Solution: Each case can be treated as an addition.

a) $2+3=5$.
b) $(-2)+(-3)=-5$.
c) $5+(-3)=2$.
d) $(+5)-(+3)=+5+(-3)=+2$.
e) $(-7)-(+6)=-7+(-6)=-13$.
f) $(-7)-(-6)=-7+(+6)=-1$.

Example 1–3. By using the fundamental assumptions and rules for operations, and transforming the left side into the right side, justify the equation

$$(a+b)-(c-d)=(b-c)+(a+d).$$

Solution: By (1–18),

$$(a + b) - (c - d) = (a + b) - c + d.$$

By (1–1) and (1–2),

$$(a + b) - c + d = (b - c) + (a + d).$$

Example 1–4. By using the fundamental assumptions and rules of operations, justify the equation

$$\frac{b}{a} \cdot \frac{ac}{b - c} = \frac{bc}{b - c}.$$

Solution: By (1–29) and (1–24),

$$\frac{b}{a} \cdot \frac{ac}{b - c} = \frac{a \cdot (bc)}{a \cdot (b - c)} = \frac{bc}{b - c}.$$

EXERCISE 1–1

1. Identify the fundamental law or laws that justify each of the following equations:

a. $x + y = y + x$.
b. $rs = sr$.
c. $2(3 \cdot 5) = (2 \cdot 3)5$.
d. $5(a + b) = 5a + 5b$.
e. $(a + 2)(b - 3) = (b - 3)(a + 2)$.
f. $(a + b)c = c(a + b) = ca + cb$.

2. Find the value of each of the following:

a. $(-3) + (+5)$.
b. $(-5) + (-3)$.
c. $(-1) - (-2)$.
d. $(+7) - (+2)$.
e. $(-8) - (-9)$.
f. $0 - (-2)$.
g. $(-5) - 0$.
h. $15 + (-3)$.
i. $(-7) - (-5)$.
j. $(-5) - (+5)$.
k. $0 + (-3) - (+4)$.
l. $(+32) - (-23) + (-45)$.

3. Evaluate each of the following:

a. $(+2)(-3)$.
b. $(-3)(-5)$.
c. $(-7)(+5)$.
d. $(-5)(-9)$.
e. $2(-5)$.
f. $(-7)0$.
g. $(-1)(-2) + (-3)(0)$.
h. $(-4)(-5) - (-2)(-1)$.
i. $(+2)(-3) - (+7)(-5)$.

4. Determine the negative of each of the following:

a. 5.
b. -3.
c. 0.
d. $2x$.
e. 2/3.
f. $2 - 3$.
g. $2a - 3b$.
h. $-(x - y)$.
i. $-[(a)(-b)]$.
j. $3x + 2$.
k. $x - 3$.
l. $a + 0$.

5. Find the reciprocal of each of the following:

a. 1.
b. 2/3.
c. $3 + \frac{2}{5}$.
d. $2 + \frac{1}{2}$.
e. 1.02.
f. $a + b$.
g. $-\frac{0.4x}{3}$.
h. $\frac{-1}{a}$.
i. $\frac{1}{x + y}$.
j. $x - 2j$.
k. $\frac{1}{r - 0.1}$.
l. $\frac{5}{2 - 0.3x}$.

6. Prove each of the following equations by using rules for signs and for operations with fractions:

a. $a \cdot (-b) + a \cdot (-c) = -a \cdot (b + c)$.

b. $-(ac - ad) = a[d + (-c)]$.

c. $-[b - (a - c)] = (a - b) - c$.

d. $b \bigg/ \dfrac{b}{a - c} + c = a$.

e. $\dfrac{a}{b} \cdot \dfrac{bc}{c - d} = \dfrac{ac}{c - d}$.

f. $c[a - (a - b/c)] = b$.

1–8. ORDER RELATIONS FOR REAL NUMBERS

We shall use the notation $a > 0$ to express the fact that a is a positive number, and the notation $a < 0$ to indicate that a is negative. The symbol $>$ means *is greater than,* and $<$ means *is less than.* These symbols are called *order symbols.*

Assume that a and b are any two given numbers. If $a - b > 0$, we shall write $a > b$, or $b < a$, and shall read "a is greater than b," or "b is less than a." As can be easily seen, $a > b$ means that a lies to the right of b on the real-number line. When $a > b$, that is, when $a - b > 0$, then $b - a$, which equals $-(a - b)$ by (1–18), is negative; and conversely. Hence, $a > b$ (or $b < a$) if and only if $b - a < 0$.

The student is familiar with the symbol $=$ (for *equality*), which is used to indicate that two quantities are the same. Thus $a = b$ means that the two symbols a and b represent the same mathematical object. For example, $6 = 3 \cdot 2$.

If a and b are two distinct numbers on the scale, we say "a is different from b" or "a does not equal b," and we write symbolically $a \neq b$. The symbol $\neq$ means *does not equal* and is called the *inequality symbol.*

In general, the oblique line or vertical line through any symbol will form a new symbol which is the negation of the original one. Thus, $a \nless b$ means "a is not less than b." In other words, $a = b$ or $a > b$ (by Property 1 below). For example, $5 \nless 3$.

Sometimes we shall find it convenient to combine the symbols $<$ and $=$ or $>$ and $=$. We write $\leqq$ to mean *is less than or equal to,* and we write $\geqq$ to mean *is greater than or equal to.*

We thus have order relations on pairs of real numbers, defined by either of the following equivalent statements:

$a > b$ (or $b < a$) if and only if $a - b$ is positive;

$a > b$ (or $b < a$) if and only if $b - a$ is negative.

The system of real numbers is then said to be *ordered* by the relation $>$ (or the relation $<$). Assertions of the type $a < b$ or $a > b$ are

called *inequalities*. The ordering of the real numbers has the following properties.

Property 1. For every pair of real numbers, a and b, one and only one of the following relationships holds:

$$a = b, \text{ or } a < b, \text{ or } a > b.$$

Proof of Property 1: If $a = b$, the statement is certainly true. Saying that $a \neq b$ is equivalent to saying that $a - b \neq 0$, so that $a - b$ is either positive or negative. Thus, if $a - b$ is positive, we have $a > b$. If, however, $a - b$ is neither positive nor zero, then it is negative, and $a < b$. If two of the three possibilities occurred together, we should have, say, $a = b$ and $a > b$, or $a > b$ and $a < b$. Thus, $a - b$ would be both zero and positive, or both positive and negative. Since no overlapping may occur among the three classes of numbers, we are thus led to a contradiction.

Property 2. For any real numbers a, b, c, it is true that

$$\text{if } a < b \text{ and } b < c, \text{ then } a < c.$$

Proof of Property 2: If $a < b$ and $b < c$, then both $b - a$ and $c - b$ are positive. Let us write $c - a$ as $(c - b) + (b - a)$. We have assumed that the sum of two positive numbers is positive. Since $c - b$ and $b - a$ are positive by assumption, their sum, which is $c - a$, is also positive. Hence, $c > a$, or $a < c$.

Property 3. For any real numbers a, b, c, it is true that

$$\text{if } a > b, \text{ then } a + c > b + c.$$

Proof of Property 3: By definition, $a > b$ means that $a - b$ is positive. But, by (1–17),

$$(a + c) - (b + c) = a + c + (-b) + (-c) = a - b.$$

It therefore follows that $(a + c) - (b + c)$ is positive, and that $a + c > b + c$.

Property 4. For any real numbers a, b, c, it is true that

$$\text{if } a > b \text{ and } c > 0, \text{ then } ac > bc.$$

Proof of Property 4: We have assumed that the product of two positive numbers is positive. Since both $a - b$ and c are positive, it follows that their product is also positive. But $(a - b)c = ac - bc$. Therefore, $ac - bc$ is positive, and $ac > bc$.

Property 5. For any real numbers a, b, c, it is true that

$$\text{if } a > b \text{ and } c < 0, \text{ then } ac < bc.$$

Proof of Property 5: We have noted that the product of two numbers with unlike signs is negative. Here $a - b$ is positive, but c is negative. Since $(a - b)c = ac - bc$, and $(a - b)c$ is negative, it follows that $ac - bc < 0$, or that $ac < bc$.

According to Property 3, the order symbol in an inequality is not changed if the same number is added to or subtracted from both sides. It therefore follows that a term on one side of an inequality may be *transposed* to the other side with its sign changed. For example, if $a - b > c$, then $a > c + b$.

According to Property 5, the order symbol in an inequality is reversed if both sides are multiplied or divided by the same negative number.

1–9. ABSOLUTE VALUE

As a consequence of the properties of the ordering of real numbers, there can be associated with each number a certain non-negative number called its *absolute value*. For any real number a, we define the absolute value of a, denoted by $|a|$, as follows:

$$|a| = a, \text{ if } a \geqq 0, \text{ and } |a| = -a, \text{ if } a < 0.$$

Thus, $|3| = 3$, since $3 > 0$; also $|-3| = -(-3) = 3$, since $-3 < 0$.

1–10. INEQUALITIES INVOLVING ABSOLUTE VALUES

We shall now consider some inequalities involving absolute values. If we let the number x be represented by a point P on a number scale, then $|x|$ is the numerical distance between P and the origin. If we let a be a positive number, then $|x| < a$ means that the point P is less than a units from the origin; that is, x lies between $-a$ and a. We can write this in the form $-a < x$ and $x < a$, or more briefly in the form $-a < x < a$. Therefore, the statements $|x| < a$ and $-a < x < a$ mean exactly the same thing.

A more general inequality which often occurs is $|x - b| < a$, where $a > 0$. This is equivalent to $-a < x - b < a$. If b is added to each term, we may write $b - a < x < b + a$. Hence, the statements $|x - b| < a$ and $b - a < x < b + a$ mean exactly the same thing.

For example, $|x - 3| < 2$ may be written $-2 < x - 3 < 2$ and means that the distance between x and 3 is less than 2. To solve this inequality for x, we add 3 to each term of the inequality, obtaining $1 < x < 5$.

The following illustrative examples may help to give a better understanding of the processes involved in the solution of the problems in Exercise 1–2.

Example 1–5. Arrange the following numbers in increasing order:

$$2,\ -3.5,\ 0,\ \pi,\ 3.14,\ |-5|.$$

Solution: Since π is approximately 3.1416, the desired order is as follows:

$$-3.5,\ 0,\ 2,\ 3.14,\ \pi,\ |-5|.$$

Example 1–6. Insert the proper inequality sign (order symbol) between the following numbers:

$$-2 \text{ and } |-2|.$$

Solution: Since $|-2| = 2$, and since $-2 < 2$, we have the inequality $-2 < |-2|$, or $|-2| > -2$.

Example 1–7. Find integers a and b such that $a < \sqrt{2} < b$.

Solution: Since $\sqrt{2}$ may be represented approximately by 1.414, the values $a = 1$ and $b = 2$ satisfy the inequalities. Thus, $1 < \sqrt{2} < 2$. Any other pair of integers a and b such that $a \leqq 1$ and $b \geqq 2$ would also satisfy the inequalities.

Example 1–8. Express the inequality $|x| < 3$ without using the absolute-value symbol.

Solution: We know that the statements $|x| < a$ and $-a < x < a$ mean exactly the same thing. Here a is the positive number 3, and $|x| < 3$ means that the point represented by x is less than 3 units from the origin; that is, x is between -3 and 3. The inequality may be written $-3 < x < 3$.

Example 1–9. Explain the meaning of the inequality $|x - 2| < 1$ and write it without using the absolute-value symbol.

Solution: The inequality $|x - b| < a$ is equivalent to $-a < x - b < a$. Hence $|x - 2| < 1$ may be written $-1 < x - 2 < 1$. If we add 2 to each term of the inequalities, we obtain $1 < x < 3$.

EXERCISE 1–2

1. Arrange the numbers in each of the following sets in increasing order:

a. $-3, 0, 4, -2, 5$.
b. $-6, -8, 2, 0, 1/2, -3/4$.
c. $-2, 10, -1, -1/3, -4$.
d. $-10, 9, 4, -3, 3/8, -6/5$.
e. $3, -2, 1, \sqrt{3}, -3/2$.
f. $1.4. 0, -2, \sqrt{2}, |-3|$.

2. Insert the proper order symbol between the two numbers in each of the following:

a. 3 and 1/3.
b. -3 and $|-3|$.
c. $\sqrt{2}$ and 1.414.
d. -3 and -2.
e. 22/7 and π.
f. 1/8 and 1/6.

3. Examine each of the following inequalities, and determine whether or not it is true.

a. $-5 > -3$.
b. $-3 + 2 < 0$.
c. $|-3| > -3$.
d. $\pi > 22/7$.
e. $|-2| < |2|$.
f. $|3 - 7| > |5 - 2|$.

4. Find the value of each of the following:
a. $|+2| - |-2|$. b. $|-3| + |+3|$. c. $|+4| - |-4|$.
d. $|-7| + |-5| - |+5|$. e. $|12 - 4| - |-6|$. f. $|5 - 3| + |3| - |2|$.
g. $(-18) \div 3$. h. $|-9| \div |4|$. i. $0 \div 14$.
j. $0|4| - |-5|$.

5. Express each of the following inequalities without using the absolute-value symbol:
a. $|x| < 1$. b. $\left|\frac{x}{2}\right| < 1$. c. $|x| \leqq a$.
d. $|2x| < 4$. e. $|x - 1| < 3$. f. $|x - 1/2| < 3/2$.

6. In each of the following, find a pair of integers, a and b, such that the given inequalities are satisfied:
a. $a < 5 < b$. b. $a < -3 < b$. c. $a < 0 < b$.
d. $a < \pi < b$. e. $a < \sqrt{3} < b$. f. $a < |1 - 2| < b$

7. If $a \leqq 3$, place the proper order symbol between $a + 7$ and 10.

8. If $a \geqq 5$, what can be said about the value of $3a - 2$?

1–11. POSITIVE INTEGRAL EXPONENTS

If two or more equal quantities are multiplied by one another, the product of the equal factors is called a *power* of the repeated factor. Thus 5^2, read "5 squared," means $5 \cdot 5$; 5^3, read "5 cubed," means $5 \cdot 5 \cdot 5$. In general, a^n means the product of n factors each equal to a. We call a the *base* and n the *exponent* of the power. It follows from the associative law that

$$a^2 \cdot a^3 = (a \cdot a)(a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5 = a^{2+3}.$$

Also, if $a \neq 0$,

$$\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a = a^2 = a^{5-3}.$$

These and similar results suggest the following laws of exponents. In them m and n are positive integers. The proofs of these laws are reserved for a later chapter.

Law of Multiplication. To multiply two powers of the same base, add the exponents:

$$a^m \cdot a^n = a^{m+n}. \tag{1–31}$$

Law of Division. To divide one power of a given base by another power of the same base, subtract the exponents:

$$\frac{a^m}{a^n} = a^{m-n}, \quad \text{if } a \neq 0, m > n. \tag{1–32}$$

Law for a Power of a Power. To raise a power of a given base to a power, multiply the exponents:

$$(a^m)^n = a^{mn}. \tag{1-33}$$

For example, $(a^3)^2 = a^3 \cdot a^3 = a^{3\cdot 2} = a^6$.

Law for a Power of a Product. To obtain a power of a product, raise each factor of the product to the given power:

$$(ab)^n = a^n b^n. \tag{1-34}$$

Thus, $(3a^3)^2 = 3^2(a^3)^2 = 3^2a^6 = 9a^6$.

Law for a Power of a Quotient. To obtain a power of a quotient, raise the numerator and the denominator to the given power:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad \text{if } b \neq 0. \tag{1-35}$$

Thus, if $b \neq 0$, $\left(\frac{a^2}{b^5}\right)^3 = \frac{(a^2)^3}{(b^5)^3} = \frac{a^6}{b^{15}}$.

1–12. ALGEBRAIC EXPRESSIONS

An algebraic expression is formed by combining numbers by means of the fundamental operations of algebra. The distinct parts of the expression connected by plus and minus signs are called *terms*. The terms of the expression $3x^2-5xy^2+7z$ are $3x^2$, $-5xy^2$, and $7z$. Here the numbers 3, −5, and 7 are called *numerical coefficients*, or just *coefficients*; x^2, xy^2, and z are called the *literal parts*.

An expression containing one or more terms is called a *multinomial*. A multinomial consisting of one term is a *monomial*. A *binomial* is a multinomial consisting of two terms, and a *trinomial* is a multinomial with three terms. A *polynomial* is a multinomial whose terms are of the form $ax^my^nz^p \cdots$, where $m, n, p, \cdots$ are positive integers and a is a numerical coefficient, and where one or more of the factors $x^m, y^n, z^p, \cdots$ may be absent. Thus, 7, $5x^4$, and $3xy + 2$ are polynomials, while $x + \frac{1}{y}$ is not.

The *degree of a term* of a polynomial is the sum of all the exponents in its literal part. For example, the degree of $3x^2$ is 2, the degree of $-5xy^2$ is 3, and the degree of $7z$ is 1, because the sums of the exponents are, respectively, 2, 3, and 1.

The *degree of a polynomial* is the degree of its *highest-degree term*. Thus, in the trinomial $3x^2 - 5xy^2 + 7z$, the third-degree term, $-5xy^2$, is its highest-degree term. Therefore, $3x^2 - 5xy^2 + 7z$ is a polynomial of the third degree.

By a polynomial in x of degree n we mean an expression of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

where the coefficients $a_0, a_1, \cdots, a_n$ are numerical coefficients, $a_0 \neq 0$, and n is a positive integer. If $n = 0$, we agree that the polynomial reduces to a number a_0, which is not 0, and that the degree is zero. The number 0 is regarded as a polynomial also, but as one having no degree.

If the typical polynomial just given has degree n, the coefficient a_0 is called the *leading coefficient.* If its leading coefficient is 1, a polynomial is called *monic.* Thus, $x^3 - 2x^2 + 5x + 7$ is a monic polynomial of degree 3.

1–13. EQUATIONS AND IDENTITIES

An *equation* is a statement of equality between two numbers or algebraic expressions. The two expressions are called *members,* or *sides,* of the equation. Equations are of two kinds, namely, *conditional equations* and *identities.* A *conditional equation,* or simply an *equation,* may be true only for certain values (possibly none at all) of the literal quantities appearing. An *identity* is true for all numerical values that can be substituted for the literal quantities.

Illustrations of equations are

$$3x - 5 = x + 1$$

and

$$x^2 - 5x + 4 = 0.$$

The first one is true only if $x = 3$, and the second is true only if $x = 1$ or $x = 4$.

Illustrations of identities are

$$3(x - 2) = 3x - 6$$

and

$$x^2 - 5x + 4 = (x - 1)(x - 4).$$

Each of these equations is true for all values of x.

1–14. SYMBOLS OF GROUPING

Parentheses ( ) and other symbols of grouping which have the same meaning as parentheses, namely, brackets [ ], braces { }, and the vinculum $\overline{\quad}$, are used to associate two or more terms which are to be combined to form a single quantity. The word "parentheses"

is often used to indicate any or all of these symbols of grouping. Removal of the symbols of grouping is accomplished by applying the laws of algebra, such as the laws of signs and the distributive law. The following examples illustrate the procedure.

Example 1–10. Remove parentheses from $-(2x - 3)$.

Solution: The steps may be indicated as follows:

$$\begin{aligned} -(2x - 3) &= (-1)(2x - 3) \\ &= (-1)(2x) + (-1)(-3) \\ &= -2x + 3. \end{aligned}$$

Example 1–11. Remove symbols of grouping from $8x - 2[5y + 3(x - y)] - \{2y - \overline{x - 3y}\}$ and collect terms.

Solution: One way of obtaining the desired result follows:

$$\begin{aligned} 8x - 2[5y + 3(x - y)] &- \{2y - \overline{x - 3y}\} \\ &= 8x - 2[5y + 3x - 3y] - \{2y - x + 3y\} \\ &= 8x - 2[2y + 3x] - \{5y - x\} \\ &= 8x - 4y - 6x - 5y + x \\ &= 3x - 9y. \end{aligned}$$

The basic rules for enclosing a group of terms in parentheses may be stated as follows:

To write a given expression in parentheses preceded by a plus sign, write the terms as they are given, enclose them in parentheses, and write + in front of the parentheses. Thus,

$$a - b = +(a - b).$$

To write a given expression in parentheses preceded by a minus sign, change the sign of each term, write the resulting terms in parentheses, and write − in front of the parentheses. Thus,

$$a - b = -(-a + b) = -(b - a).$$

The first rule is obvious, and the second follows from the rule of signs (1–18).

Example 1–12.

a) Enclose the last two terms of $2 + 3x - y$ within parentheses preceded by a plus sign.

b) Enclose these terms within parentheses preceded by a minus sign.

Solution:

a) Since the sign before the parentheses is to be +, we enclose $3x - y$ in its given form within the parentheses preceded by a plus sign. In this case the expression becomes $2 + (3x - y)$.

b) Since the sign before the parentheses is to be $-$, we change the sign of each term of $3x - y$ and enclose $-3x + y$ within the parentheses preceded by a minus sign. Then the expression becomes $2 - (-3x + y)$.

1–15. ORDER OF FUNDAMENTAL OPERATIONS

Parentheses and other symbols of grouping are useful in indicating which operation is to be performed first. We have used them in this way from the outset. In order to avoid using them unnecessarily, as has been already pointed out, the convention is adopted to perform all multiplications first and then the additions (or subtractions). If two or more of these symbols of grouping are used in the same expression, we usually (though not necessarily) remove the innermost pair of symbols first.

Illustrations in which symbols of grouping are removed follow:

a) $4 - (6/2) = 4 - 3 = 1.$
b) $(4 - 6)/2 = (-2)/2 = -1.$
c) $6 + [15/(3 \cdot 5)] = 6 + (15/15) = 6 + 1 = 7.$
d) $6 + (15/3) \cdot 5 = 6 + 5 \cdot 5 = 6 + 25 = 31.$

EXERCISE 1–3

In each problem in the group from 1 to 36, perform the indicated operation or operations.

1. 2^5. **2.** 3^4. **3.** $(-1)^3$. **4.** 5^3.
5. 10^6. **6.** $(-2)^4$. **7.** $(-6)^3$. **8.** -10^5.
9. 7^4. **10.** $(-4)^3$. **11.** $\left(\frac{1}{2}\right)^4$. **12.** $\left(-\frac{2}{3}\right)^3$.
13. $\left(\frac{4}{7}\right)^7$. **14.** $\left(\frac{2}{9}\right)^3$. **15.** $\left(-\frac{3}{4}\right)^4$. **16.** $4(2^6)$.
17. $\frac{1}{2}(4^3)$. **18.** $\frac{2}{3}(3^4)$. **19.** $a(5^3)$. **20.** $a^2b(-2)^3$.
21. a^3a^4. **22.** $(a^3)^4$. **23.** $(ab)^4$. **24.** a^4a^7.
25. $(a^2b^3)^2$. **26.** $a \cdot a^2 \cdot a^3$. **27.** $a^2 \cdot a^4 \cdot a^7$. **28.** $[(ab)^n]^2$.
29. $(a^nb^m)^2$. **30.** $(a^2b^2)^n$. **31.** $\frac{a^6}{a^2}$. **32.** $\frac{a^7}{a^3}$.
33. $\frac{a^4}{a^8}$. **34.** $\left(\frac{a}{4}\right)^3$. **35.** $\left(\frac{2}{a}\right)^4$. **36.** $\left(\frac{3^2}{a^3}\right)^5$.

In each problem in the group from 37 to 48, remove symbols of grouping and simplify.

37. $3 - (b - 2)$. **38.** $x + (y - z)$.
39. $4[a + (b - a)]$. **40.** $(a + 2b) - (3a - b)$.
41. $a - b + (2a - 3b)$. **42.** $a - [3 + (2a - 4)]$.
43. $(a - 2b) - (3a + b)$. **44.** $6x^2 - [3x + (2y - x^2)]$.

45. $ax^2 - 2bxy + [(by^2 - 2cx^2) - (axy + y^2)]$.
46. $3ab - \{4ac + [ab - \overline{2ac + ab}] - 3ab\}$.
47. $2a - [3b + 4c - \{3a - b + \overline{a - b} - (3a + 2c)\} - \overline{4c + a}]$.
48. $-\{-[-(a - \overline{b - c}) - a + (b - c)]\}$.

In each problem from 49 to 60, enclose the last two terms in parentheses. First use a plus sign before the parentheses, and then use a minus sign.

49. $a + b + c$.
50. $a^2 - 2ab + b^2$.
51. $a^2 - b^2 + c^2$.
52. $x + y - 1$.
53. $2a + b - 3c$.
54. $3x - 4y + 2z$.
55. $x^2 - y^2 - z^2$.
56. $-x^2 - y^2 - z^2$.
57. $-a^3b + ab + b^2$.
58. $ax^2 - 2axy + y^2$.
59. $2x - 3y - 4z$.
60. $-x^2 - x + 1$.

In each of the following problems, evaluate the given expression.

61. $16 - (6 - 2)$.
62. $16 - 6 - 2$.
63. $(-3)(-4) - (4)(-2)$.
64. $4 \cdot (6 - 7)$.
65. $4 \cdot 6 - 7$.
66. $(4 \cdot 6) - 7$.
67. $(3 \cdot 3) - (4 \cdot 2)$.
68. $3 \cdot 3 - 4 \cdot 2$.
69. $3 \cdot (3 - 4) \cdot 2$.
70. $3 \cdot 3 - (4 \cdot 2)$.
71. $(8/2) + 4$.
72. $8/(2 + 4)$.
73. $8 + (4/2)$.
74. $(8 + 4)/2$.
75. $8 + 4(1/2) + 3$.
76. $(8 + 4)/(2 + 3)$.
77. $8 + (4/2) + 3$.
78. $8 + ((4/2) + 3)$.
79. $\dfrac{3(6 - 4)}{3 \cdot 6 - 4}$.
80. $\left(\dfrac{3(6 - 4)}{3 \cdot 6 - 4}\Big/3\right) - 1$.
81. $\dfrac{3(6 - 4)}{3 \cdot 6 - 4}\Big/(3 - 1)$.
82. $\left[\dfrac{3(6 - 4)}{3 \cdot 6 - 4} - 1\right]\Big/3$.

1–16. ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS

Terms, such as $2x^2y^3$ and $5x^2y^3$, which have the same literal parts, are called *similar* or *like* terms and may be added or subtracted by adding or subtracting their coefficients. To illustrate, let us consider an example.

Example 1–13. Add $5xy^2, 7x^2y, -2xy^2, -9x^2y, 4x^2y^3$.

Solution: Collecting like terms and adding coefficients, we have

$$(5 - 2)xy^2 + (7 - 9)x^2y + 4x^2y^3 = 3xy^2 - 2x^2y + 4x^2y^3.$$

The procedure used in the solution of Example 1–13 follows at once from the distributive law. For example, the sum of the terms $5xy^2$ and $-2xy^2$ is obtained as $(5 - 2)xy^2$ or $3xy^2$.

This leads at once to the rule for the addition (or subtraction) of algebraic expressions. In practice, we usually arrange like terms in vertical columns, and then we find the sum of each column by prefixing the sum of the numerical coefficients in the column. The procedure may be made clearer by means of the following examples.

Example 1–14. Add $2x^2 - 3xy + z$, $x^2 - 5z$, $2xy + 3z$.

Solution:

$$\begin{array}{r} 2x^2 - 3xy + z \\ x^2 \qquad\quad - 5z \\ \underline{2xy + 3z} \\ 3x^2 - xy - z. \end{array}$$

Example 1–15. Subtract $2a^2 - 3b + c^2$ from $3a^2 - c^2$.

Solution:

$$\begin{array}{l} 3a^2 \qquad\quad - c^2 \\ \underline{2a^2 - 3b + c^2} \\ \;a^2 + 3b - 2c^2. \end{array}$$

1–17. MULTIPLICATION OF ALGEBRAIC EXPRESSIONS

With the help of the distributive law for multiplication, the product of two algebraic expressions is found by multiplying each term of one by each term of the other and combining like terms.

Example 1–16. Multiply $3x^2 - 2xy + y^2$ by $2x - 3y$.

Solution:

$$\begin{array}{l} 3x^2 - 2xy + y^2 \\ \qquad\;\, 2x - 3y \\ \hline 6x^3 - 4x^2y + 2xy^2 \\ \qquad - 9x^2y + 6xy^2 - 3y^3 \\ \hline 6x^3 - 13x^2y + 8xy^2 - 3y^3. \end{array}$$

1–18. SPECIAL PRODUCTS

The following typical forms of multiplication occur so frequently that we should learn to recognize them quickly and to obtain the products without resorting to the general process of multiplication. They should be learned thoroughly.

$$a(b + c) = ab + ac. \tag{1–5}$$

$$(a + b)(a - b) = a^2 - b^2. \tag{1–36}$$

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3. \tag{1–37}$$

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3. \tag{1–38}$$

$$(a + b)^2 = a^2 + 2ab + b^2. \tag{1–39}$$

$$(a - b)^2 = a^2 - 2ab + b^2. \tag{1–40}$$

$$(ax + by)(cx + dy) = acx^2 + (ad + bc)xy + bdy^2. \tag{1–41}$$

1–19. DIVISION OF ALGEBRAIC EXPRESSIONS

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and add the results.

This rule follows immediately from the rule for fractions expressed by (1–25).

Example 1–17. Divide $6a^2x^2 - 12a^4x - 30a^6x^5$ by $15a^3x^2$.

Solution:
$$\frac{6a^2x^2 - 12a^4x - 30a^6x^5}{15a^3x^2} = \frac{6a^2x^2}{15a^3x^2} - \frac{12a^4x}{15a^3x^2} - \frac{30a^6x^5}{15a^3x^2}$$
$$= \frac{2}{5a} - \frac{4a}{5x} - 2a^3x^3.$$

To divide a polynomial (the *dividend*) by a polynomial (the *divisor*), arrange both according to descending or ascending powers of some common literal quantity. Then proceed as follows:

Divide the first term of the dividend by the first term of the divisor to obtain the *first* term of the quotient.

Multiply the entire divisor by the first term of the quotient, and subtract this product from the dividend.

Use the *remainder* found by this process as a *new dividend,* and repeat the process. Continue the work until you obtain a remainder that is of *lower degree* in the common literal quantity than the divisor.

Example 1–18. Divide $6x^3 - 5x^2 + 3x + 1$ by $2x - 1$.

Solution:
$$\begin{array}{r|l|l}
\underline{2x - 1} & 6x^3 - 5x^2 + 3x + 1 & \underline{3x^2 - x + 1} \\
 & \underline{6x^3 - 3x^2} & \\
 & \quad - 2x^2 + 3x & \\
 & \quad \underline{- 2x^2 + x} & \\
 & \qquad\quad 2x + 1 & \\
 & \qquad\quad \underline{2x - 1} & \\
 & \qquad\qquad\quad 2. &
\end{array}$$

The division can be checked by finding the product of $(2x - 1)$ and $(3x^2 - x + 1)$ and adding 2, proving that $6x^3 - 5x^2 + 3x + 1 = (2x - 1)(3x^2 - x + 1) + 2.$

Any problem in division, in general, may be checked by means of the relationship

dividend = (divisor) (quotient) + remainder.

This equation is an identity; that is, it is true for all values of the literal quantities. Indeed, this equation supplies the underlying meaning of the process of division.

EXERCISE 1–4

In each problem in the group from 1 to 12, add the given expressions which are separated by commas.

1. $-xy, 3xy.$ **2.** $4x^2y^2, -2xy.$ **3.** $3x^2y^2, -2x^2y^2.$
4. $6a + 11b, -3a + 2b.$ **5.** $4a^2 - 2b^2, a^2 - 4b^2.$ **6.** $3a - 2b + 4c, 4a + 3b - 6c.$
7. $-3a + b - c, -a - b + c.$
8. $3x + 2y - z, x + y - 3z, 4x - 3y + 2z.$
9. $x^2 - 2xy + y^2, 4xy, -y^2.$
10. $3x^2 - 4xy^2 + 2y^3, 4x^2y - 2x^2 - y^3, 4xy^2 - 2y^3.$
11. $2x^3 - 3x + 1, x^2 + 2x - 3, 2x^2 - x^3 + 4 - 2x.$
12. $4ax + 3bxy - 4x^2, 2bx - bx^2 + 2axy, 3x - 2y.$

In each problem from 13 to 24, subtract the second expression from the first.

13. $-xy, 3xy.$ **14.** $4x^2y^2, -2xy.$
15. $3x^2y^2, -2x^2y^2.$ **16.** $6a + 11b, -3a + 2b.$
17. $4a^2 - 2b^2, a^2 - 4b^2.$ **18.** $3a - 2b + 4c, 4a + 3b - 6c.$
19. $-3a + b - c, -a - b + c.$ **20.** $x^3 + x^2 + x + 1, x^3 - x^2 - x - 1.$
21. $x^2 + 2xy + y^2, 4xy.$
22. $3 - 2x + x^2 - x^3 + 3x^4 - 2x^5, 3x^5 + 4x^4 - x^3 + 2x^2 - 2x + 1.$
23. $\frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{2}{7}xy - \frac{2}{3}y^2, \frac{2}{5}x^3 + \frac{1}{4}xy - \frac{7}{2}y^2.$
24. $\frac{1}{6}xy^2 - \frac{1}{7}x^2y + \frac{1}{4}xy^2 - \frac{1}{4}y^2, \frac{1}{4}x^2y - \frac{1}{5}xy + \frac{2}{3}y^2.$

In each problem from 25 to 60, perform the indicated operations.

25. $(6x)(-3y).$ **26.** $(4x)(-2).$
27. $(5x^2y)(-2xy).$ **28.** $(3a)(2b^2)(-4c).$
29. $(-3ab)(4bc)(-2a^3).$ **30.** $(x - y)2a.$
31. $(3x + 2y)(-2x).$ **32.** $(6 - 3a)(2).$
33. $(4xy + 2y^2)(3xy).$ **34.** $(4a^2 + 6b^2)(3ab).$
35. $(2x + 3y)(x - y).$ **36.** $(3a - b)(a + 2b).$
37. $4x^2(x^2 - 2xy - y^2).$ **38.** $3xy(2x^2y + 3xy - 4y^2).$
39. $(x^2 + xy - y^2)(x - y).$ **40.** $(x^2 - y^2)(x + y)(x - y).$
41. $4x/(-2).$ **42.** $3x^2y/2xy.$
43. $4x^2y^3z/2xy^2z.$ **44.** $20x^6y^4z^9/10x^5yz^3.$
45. $4ab/(-4ab).$ **46.** $(2xy)^2/2xy^2.$
47. $(2xy)^2/2(xy)^2.$ **48.** $(x^2 - 2xy)/(-x).$
49. $(3xy^2 - 6xy + 9y^2)/(-3y).$ **50.** $(x^2y^3 - 3x^2y^2 + 4x^3y^2)/x^2y^2.$
51. $(x^2 + 6x + 5)/(x + 1).$ **52.** $(9x^2 - 6x + 1)/(3x - 1).$
53. $(x^2 - y^2)/(x - y).$ **54.** $(x - y)^2/(x - y).$
55. $(x^4 - y^4)/(x - y).$ **56.** $(x^3 + 3x^2 + 3x + 1)/(x + 1)^2.$
57. $(x^3 + y^3)/(x + y).$ **58.** $(x + y)^3/(x + y).$
59. $(x^3 - y^3)/(x - y).$ **60.** $(x^4 + x^3 + 3x^2 + 2x + 1)/(x^2 + 2).$
61. Divide $x^2 - y^2 - (x - y)^2 - (x - y)(x\,y)$ by $x - y$.
62. Divide $(a + b)^2 + 6(a + b) + 5$ by $a + b + 5$.
63. Divide $x^4 + 4x^3 + 6x^2 + 4x + 1$ first by $x + 1$ and then by $x^2 + 2x + 1$.
64. Multiply $x^2 - 2x - 3$ by $x + 4$, and divide the result by $x + 1$.

65. Divide $x^5 + x^4 + 3x^3 - 2x^2 - 3$ by $x - 1$ and add the quotient to the excess of $3x^4 + 2x^2 - 9x + 7$ over $2x^4 + 2x^3 + 7x^2 + 3x + 4$.
66. *a*) Under what conditions will $(-x)^n$ be positive? *b*) When will it be negative? Assume first that x is positive and then that x is negative.
67. For what values of n will $(-a)^n$ be equal to $-a^n$?

1–20. FACTORING

Factoring a quantity is the process of finding quantities which, when multiplied together, yield the given quantity. When a quantity A is expressed as a product $B \cdot C$, B and C are called *factors* or *divisors* of A and are said to *divide* A. Also, A is called a *multiple* of each of B and C. These concepts are applicable to numbers or algebraic expressions generally, but are most useful when restricted to apply to integers or to polynomials. Such restriction will be adhered to in this book. Thus, when an integer is to be factored, the factors sought are to be integers. And when a polynomial is to be factored, the desired factors are to be polynomials.

Let us first review the fundamentals of factoring integers (positive, negative, or zero). First, it is clear that every integer n may be expressed as $1 \cdot n$ or $(-1) \cdot (-n)$. Such factorizations are called *trivial.* If an integer n, other than $+1$ or -1, has no factorizations other than trivial ones, then n is called a *prime* (number). An integer having a non-trivial factorization is called *composite.* Examples of prime integers are 3, 7, and -11; examples of composite integers are 6 and -40.

Let n be a composite integer. Then a non-trivial factorization $m \cdot p$ exists in which $|m|$ and $|p|$ are less than $|n|$ and greater than 1. If both m and p are primes, then n is expressed as a product of primes. If not, at least one of m and p, say p, is composite, and so $p = r \cdot s$. Hence, $n = m \cdot r \cdot s$. The process begun may be continued if any one of m, r, and s is composite, and additional factors may be found, until the process cannot be continued further, in which case only prime factors are obtained. We may conclude that the factoring process must terminate, since at any stage the new factors introduced are numerically less than their product. The *fundamental theorem of arithmetic* guarantees that every composite integer is a product of primes, which are unique except for their signs or the order in which they are written. For example, successive factoring of 156 gives

$$156 = 39 \cdot 4 = 13 \cdot 3 \cdot 4 = 13 \cdot 3 \cdot 2 \cdot 2,$$

and the final factors 13, 3, 2, 2 are primes. Another valid factorization of 156 into primes is as follows:

$$156 = 2 \cdot 3 \cdot (-13) \cdot (-2).$$

Let us turn now to polynomials with real coefficients. These may be polynomials in x, such as $x^2 - 1$; or polynomials in x and y, such as $x^2 + 3xy + 2y^2$; or, in fact, polynomials in any number of literal quantities. Every polynomial F has a factorization of the form

$$F = \frac{1}{a} \cdot (aF),$$

for every non-zero number a. And, of course, the factors $1/a$ and aF are themselves polynomials. Such factorizations are called *trivial.* If a polynomial F has no factorizations other than trivial ones, then F is called a *prime* polynomial, or an *irreducible* polynomial. A polynomial having a non-trivial factorization is called *composite* or *reducible.*

Examples of prime polynomials are $3x + 2$, $2x + 2y$, and $x^2 + 3$. Every polynomial in x of the first degree may be shown to be prime; and certain polynomials of higher (even) degree in x also are prime. A development of criteria for primeness of polynomials lies beyond the scope of this book.

Examples of composite polynomials are $x^2 - 4$, $xy + y^2$ and $xz + yz + xu + yu$, because each of these has a non-trivial factorization. Thus, we have

$$\begin{aligned} x^2 - 4 &= (x + 2)(x - 2), \\ xy + y^2 &= (x + y)y, \\ xz + yz + xu + yu &= (x + y)(z + u). \end{aligned}$$

Let f be a composite polynomial in x with real coefficients, not all of which are zero. Then it can be shown that there exist polynomials g and h, the degree of each of which is less than that of f, such that

$$f = g \cdot h.$$

If g and h are primes, then f is a product of primes. If not, we may proceed to factor (non-trivially) one or both of g and h, and we can continue this process until primes are obtained. The process must terminate eventually, since each non-trivial factorization leads to polynomials of lower degree.

The argument just presented applies only to polynomials in one literal quantity x. However, the principle may be extended to apply to polynomials in any number of literal quantities x, y, z, $\cdots$. Thus,

$$f = p_1 p_2 \cdots p_n,$$

where $p_1, p_2, \cdots, p_n$ are prime polynomials. The problem of carry-

ing out actual factorizations for certain types of polynomials is considered in the next section.

A special class of polynomials deserves particular attention. This is the class that consists of polynomials in which all the numerical coefficients are integers. It is possible to prove a factorization theorem such as that just stated, but yielding factors which are polynomials having only integral coefficients. When the general theorem can yield prime factors all of which have integral coefficients, the two theorems give the same result. Otherwise, they will give different results.

For example, both theorems applied to $x^2 - 4$ yield the prime factorization $(x + 2)(x - 2)$. The polynomial $3x^2 - 4$ has no nontrivial factors with integral coefficients. However, when other real coefficients are allowed, we have

$$3x^2 - 4 = (\sqrt{3}x + 2)(\sqrt{3}x - 2),$$

in which the coefficient $\sqrt{3}$ is a perfectly acceptable real number. Again, when factors with integral coefficients are desired in such a case as $36x + 24y$, we shall agree to remove and factor common numerical factors, to obtain

$$36x + 24y = 3 \cdot 2 \cdot 2 \cdot (3x + 2y).$$

Here the prime factors are the numerical primes 3, 2, 2 and the prime polynomial $3x + 2y$.

In what follows, whenever a polynomial has only integral coefficients, we agree to restrict ourselves to factors of the same kind. In similar fashion, when the given polynomial has only rational coefficients, we shall search for factors having only rational coefficients.

1–21. IMPORTANT TYPE FORMS FOR FACTORING

The equations in Section 1–18 applied "in reverse" are formulas for factoring. Success in factoring a polynomial therefore depends on ability to recognize the polynomial as being a particular type of product and as having factors of a definite form. Verify the following type forms by carrying out the indicated multiplications and learn each form.

Type 1: Common Monomial Factor. From (1–5) we have

(1–42) $$ab + ac = a(b + c).$$

Example 1–19. Factor $4x^2y - 6xy^2$.

Solution: $4x^2y - 6xy^2 = 2xy(2x - 3y)$.

Instructions for actually removing the monomial factor in Example 1–19 may be put this way: Write the common factor, $2xy$, and in parentheses following $2xy$ write the algebraic sum of the quotients obtained by dividing successively every term of $4x^2y - 6xy^2$ by $2xy$, in accordance with the distributive law.

Type 2: Difference of Two Squares. From (1–36) we have

$$a^2 - b^2 = (a + b)(a - b). \tag{1-43}$$

Example 1–20. Factor $9x^2 - 25y^2$.

Solution: $9x^2 - 25y^2 = (3x + 5y)(3x - 5y)$.

Type 3: Sum and Difference of Two Cubes. From (1–37) and (1–38) we have

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \tag{1-44}$$

and

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \tag{1-45}$$

Example 1–21. Factor $8x^3 - 27y^3$.

Solution:
$$\begin{aligned} 8x^3 - 27y^3 &= (2x - 3y)[(2x)^2 + (2x)(3y) + (3y)^2] \\ &= (2x - 3y)(4x^2 + 6xy + 9y^2). \end{aligned}$$

Type 4: Perfect-Square Trinomials. From (1–39) and (1–40),

$$a^2 + 2ab + b^2 = (a + b)^2 \tag{1-46}$$

and

$$a^2 - 2ab + b^2 = (a - b)^2. \tag{1-47}$$

Example 1–22. Factor $4x^2 + 12xy^2 + 9y^4$.

Solution:
$$\begin{aligned} 4x^2 + 12xy^2 + 9y^4 &= (2x)^2 + 2(2x)(3y^2) + (3y^2)^2 \\ &= (2x + 3y^2)^2. \end{aligned}$$

Type 5: General Trinomial. Factorization of the general trinomial may be indicated as follows:

$$Ax^2 + Bxy + Cy^2 = (ax + by)(cx + dy). \tag{1-48}$$

If the two factors $(ax + by)$ and $(cx + dy)$ are multiplied together, the product is found to be

$$acx^2 + (bc + ad)xy + bdy^2.$$

By comparing this product with the trinomial

$$Ax^2 + Bxy + Cy^2,$$

we note that it is necessary to find four numbers a, b, c, d, such that

$$ac = A,\ bc + ad = B, \text{ and } bd = C.$$

The following example illustrates a trial-and-error procedure, which often involves several steps of inspection and testing by multiplication; yet it is a method commonly employed in practical work and is recommended here.

Example 1–23. Factor $6x^2 + 11xy - 10y^2$.

Solution: Here we wish to find a, b, c, and d which satisfy the identity

$$(ax + by)(cx + dy) = acx^2 + (ad + bc)xy + bdy^2 = 6x^2 + 11xy - 10y^2.$$

Since $ac = 6$ and $bd = -10$, obviously a and c have like signs, and b and d have opposite signs. Possible values for a and c are ± 1, ± 2, ± 3, and ± 6. Possible values for b and d are ± 1, ± 2, ± 5, and ± 10. By trial and error we find the correct selection to be $a = 2$, $c = 3$, $b = 5$, and $d = -2$. This selection meets the requirement because

$$(2x + 5y)(3x - 2y) = 6x^2 + 11xy - 10y^2.$$

Type 6: Factoring by Grouping. An expression which does not fall directly into one of the given type forms can sometimes be reduced to one of these forms by a suitable grouping of terms. The following examples illustrate the procedure.

Example 1–24. Factor $3ax - 5bx + 6ay - 10by$.

Solution: First, group within parentheses the terms having a common factor. Thus,

$$3ax - 5bx + 6ay - 10by = (3ax - 5bx) + (6ay - 10by).$$

Then, in each group, factor out the common quantity. In this case,

$$(3ax - 5bx) + (6ax - 10by) = x(3a - 5b) + 2y(3a - 5b).$$

Finally, factor out the quantity common to the terms obtained. (This quantity will often be a multinomial factor.) The result is

$$x(3a - 5b) + 2y(3a - 5b) = (3a - 5b)(x + 2y).$$

An alternate method of grouping would give us the following results: First,

$$3ax - 5bx + 6ay - 10by = (3ax + 6ay) + (-5bx - 10by).$$

Then,

$$(3ax + 6ay) + (-5bx - 10by) = 3a(x + 2y) - 5b(x + 2y).$$

Finally,

$$3a(x + 2y) - 5b(x + 2y) = (x + 2y)(3a - 5b).$$

Example 1–25. Factor $x^2 - y^2 + 6yz - 9z^2$.

Solution: By grouping the last three terms, we may rewrite the given expression as the difference of two squares. Thus,

$$\begin{aligned} x^2 - y^2 + 6yz - 9z^2 &= x^2 - (y^2 - 6yz + 9z^2) \\ &= x^2 - (y - 3z)^2 \\ &= [x - (y - 3z)][x + (y - 3z)] \\ &= (x - y + 3z)(x + y - 3z). \end{aligned}$$

Example 1–26. Factor $4x^4 + 3x^2y^4 + y^8$.

Solution: By adding and subtracting x^2y^4, we may rewrite the given expression as the difference of two squares. Thus,

$$\begin{aligned} 4x^4 + 3x^2y^4 + y^8 &= 4x^4 + 4x^2y^4 + y^8 - x^2y^4 \\ &= (2x^2 + y^4)^2 - x^2y^4 \\ &= [(2x^2 + y^4) - xy^2]\,[(2x^2 + y^4) + xy^2] \\ &= (2x^2 + y^4 - xy^2)\,(2x^2 + y^4 + xy^2). \end{aligned}$$

EXERCISE 1–5

Factor each of the following:

1. $3x + 6y$.
2. $6x^2 + 4xy$.
3. $4x + 14$.
4. $2a - ab$.
5. $3ax + 2a$.
6. $4ax^2 - ya^2x^3$.
7. $-ax + 2cx - x^2$.
8. $2at - 3bt^2 + 6ct^3$.
9. $ax - 2ay + 3az$.
10. $3ab + 9ac - 6bc$.
11. $5y^2 + 3y^3 - ay^2$.
12. $x^3 - 5x^2y + 6xy^2$.
13. $x^2 - 16$.
14. $y^2 - 64$.
15. $4x^2 - 9$.
16. $16x^4 - 25y^4$.
17. $49x^2 - 121$.
18. $25a^2 - 16b^2$.
19. $9y^2 - a^2$. ✓
20. $1 - x^2y^2$.
21. $a^2x^4 - 9a^4x^2$.
22. $4a^2x^4 - 81$.
23. $0.01 - b^2$.
24. $-a^3x^2 - ax^3$.
25. $49x^2y^2 - 144a^2b^2$.
26. $a^2 - \frac{1}{9}$.
27. $36x - x^5$.
28. $4x^2 - 16x^4$.
29. $2x^3 + 16$.
30. $3x^3 - 81y^3$.
31. $x^3y^3 + z^6$.
32. $a^6 - 216b^9$.
33. $125p^6q^9 + r^{15}$.
34. $a^3 + b^9$.
35. $x^6 - y^{12}$.
36. $2x^6y^2 - 128y^2$.
37. $81x^6 - 3(2y + 3z)^3$.
38. $(a + 2b)^3 + (3c + 4d)^3$.
39. $(x + y)^3 - (z - w)^3$.
40. [illegible] $- y^6$.
41. $216x^3 - y^3$.
42. $x^2 - 12x + 36$.

[illegible] $- 12x^3 + x^5$.

44. $8x^2 + 2x - 15$.
45. $x^2y^2 - 18xy + 81$.

[illegible] $x - 6$.

47. $x^4 - 8x^2 + 15$.
48. $x^2 - 11x + 30$.

[illegible] $8x^2 - 2x - 3$.

50. $2x^2 - 3x + 1$.
51. $x^2 - 4xy + 4y^2$.
52. $x^4 - 2x^2 + 1$.
53. $x^2 - x - 12$.
54. $x^2 - 3x - 10$.
55. $x^4 + 3x^2 + 2$.
56. $x^6 + 4x^3 + 4$.
57. $1 + 30x^3 + 225x^6$.
58. $x^4 - 10x^2 + 9$.
59. $5x^3 + 10x^2 - 40x$.
60. $a^2 + 5a - 24$.
61. $2x^2 - 11x - 6$.
62. $x^2 + 16x + 64$.
63. $2x^2 + 5x - 3$.
64. $18x^2 + 15x - 25$.
65. $6x^2 - 37x + 6$.
66. $4x^2 + 32x + 15$.
67. $x^2 - 1.2x + 0.36$.
68. $40x^2y^2 + 35xyz - 18z^2$.
69. $ax + 3x + 2ay + 6y$.
70. $x^3 + 3x^2 - 7x - 21$.
71. $8x^3 - 12x^2 - 10x + 15$.
72. $ax + 2bx - cx + 3ay + 6by - 3cy$.
73. $2ax + 3y - xz + 6a + xy - 3z$.
74. $xab - xyz - 2aby + 2y^2z$.
75. $x^3 + x^2 - 3x - 3$.

1–22. GREATEST COMMON DIVISOR

A *common divisor* of several polynomials (or integers) is a polynomial (or integer) which divides each of them. For example, 2 and 3 are common divisors of 12 and 18. Also, x and $x + y$ are common divisors of $x^2(x^2 - y^2)$ and $x^3(x^2 + 2xy + y^2)$.

A *greatest common divisor* (G.C.D.), also known as a *highest common factor* (H.C.F.), of two or more polynomials (or integers)

is a polynomial (or integer) with the following two properties: It is a common divisor of the given polynomials (or integers); also it is a multiple of every other common divisor of the given polynomials (or integers).

It follows that, for integers, a G.C.D. is a common divisor of greatest absolute value. It also follows that, for polynomials, a G.C.D. is a common divisor of highest degree.

For example, a G.C.D. of 12 and 18 is $+6$ or -6, since ± 1, ± 2, ± 3, and ± 6 are the only common divisors, and 6 and -6 are those of maximum absolute value. This example indicates that a set of non-zero integers will have two greatest common divisors, d and $-d$, one being positive and the other negative.

For polynomials with real coefficients, if d is a G.C.D., then $a \cdot d$ is also a G.C.D. for every real number a not equal to 0. It follows that infinitely many greatest common divisors exist. However, for polynomials with integral coefficients, a G.C.D. should be a polynomial of the same type. Example 1–27, which follows, illustrates the fact that in this case a G.C.D. is uniquely determined except for sign.

Since a G.C.D. of given polynomials divides all of them, it must contain as a factor each of the distinct prime factors occurring as a *common* factor of all the given polynomials. Since, however, a G.C.D. must divide any common factor of the polynomials, it must contain each of the distinct common primes raised to the highest common power. Thus, a G.C.D. of $x^3 (x + y)^2$ and $x^5 (x - y) (x + y)$ must contain the prime factor x to the third power, that is, to the highest common power. Similarly, it must contain $(x + y)$ to the first power. Thus, $x^3(x + y)$ is a G.C.D., since no further common prime factors occur.

When no common prime factors occur, a G.C.D. is 1.

Example 1–27. Find a G.C.D. of $3x^2y^3(x^2 - 4y^2)$ and $6xy^2(x^2 - 4xy + 4y^2)$.

Solution: We shall begin by writing each of the expressions as the product of its prime factors, as follows:

$$3x^2y^3(x^2 - 4y^2) = (3)(x)(x)(y)(y)(y)(x + 2y)(x - 2y),$$

and

$$6xy^2(x^2 - 4xy + 4y^2) = (2)(3)(x)(y)(y)(x - 2y)(x - 2y).$$

The different prime factors are 2, 3, x, y, $(x + 2y)$, and $(x - 2y)$, of which only 3, x, y, and $x - 2y$ are common to both polynomials. We now form the product of these common factors, using for each the maximum common power. Hence, a G.C.D. is

$$3xy^2(x - 2y).$$

1–23. LEAST COMMON MULTIPLE

A *common multiple* of two or more polynomials (or integers) is one containing each of the given ones as a factor. Thus, 36 is a common multiple of 6 and 9, and $x^2 - y^2$ is a common multiple of $x - y$ and $x + y$.

A *least common multiple* (L.C.M.) of two or more polynomials (or integers) is a polynomial (or integer) with the following properties: It is a common multiple of the given polynomials (or integers); also it is a divisor of every other common multiple of the given polynomials (or integers).

It follows that, for integers, an L.C.M. is a common multiple of least absolute value. It also follows that, for polynomials, an L.C.M. is a common multiple of lowest degree.

For example, an L.C.M. of 6 and -8 is 24 or -24, since the only common multiples are ± 24, ± 48, $\pm 72, \cdots$, and 24 and -24 are those of minimum absolute value. This example indicates that a set of non-zero integers will have two least common multiples, m and $-m$, one being positive and the other negative.

For polynomials with real coefficients, if m is an L.C.M., then $a \cdot m$ is also an L.C.M. for every real number a not equal to 0. It follows that infinitely many least common multiples exist. However, for polynomials with integral coefficients, an L.C.M. should be a polynomial of the same type. Example 1–28, which follows, illustrates the fact that in this case an L.C.M. is uniquely determined except for sign.

Since an L.C.M. of given polynomials is a multiple of all of them, it must contain as a factor each of the distinct prime factors occurring as a factor of *any one* of the given polynomials. Since, however, an L.C.M. must be a multiple of any common multiple of the given polynomials, it must contain each of the various distinct primes to the highest power occurring anywhere. For example, an L.C.M. of $x^3(x + y)^2$ and $x^5(x - y)(x + y)$ must contain the various distinct prime factors, which are x, $x + y$, and $x - y$. For x, the highest power occurring anywhere is the fifth power; for $x + y$, the highest power is the second; for $x - y$, the highest power is the first. An L.C.M. is therefore $x^5(x + y)^2(x - y)$.

Example 1–28. Find an L.C.M. of $4x^2 - 4x$, $6x^2 - 6$, and $9x^2 - 18x + 9$.

Solution: We shall first rewrite each of the expressions in factored form. Thus,

$$4x^2 - 4x = (2)(2)(x)(x - 1) = 2^2x(x - 1),$$
$$6x^2 - 6 = (2)(3)(x + 1)(x - 1),$$

and

$$9x^2 - 18x + 9 = (3)(3)(x - 1)(x - 1) = 3^2(x - 1)^2.$$

The distinct prime factors are 2, 3, x, $x + 1$, and $x - 1$. The greatest powers for these are 2, 2, 1, 1, and 2, respectively. An L.C.M. is therefore

$$2^2 \cdot 3^2 \cdot x \cdot (x + 1) \cdot (x - 1)^2.$$

EXERCISE 1–6

In each problem from 1 to 12, find a G.C.D. of the given expressions.

1. 4, 14, 36. **2.** 9, 21, 33. **3.** 4, 7, 39.

4. $x + y$, $x^2 - y^2$. **5.** $x^3 - y^3$, $x - y$. **6.** $3ab$, $12a^3b$, $6a^3b^2$.

7. $9x^3y^2$, $15x^4y^2$, $21x^6y$. **8.** $x - 3$, $x^2 - 9$, $(x - 3)^2$. **9.** $4x^5y^4z$, $8xy^7z^6$, $14x^3y^3z^2$.

10. $x + 2$, $x^2 - 4$, $x^3 + 2x^2 - 4x - 8$.

11. $x^3 - x^2 - 42x$, $x^4 - 49x^2$, $x^2 - 36$.

12. $x^4 + 2x^3 - 3x^2$, $2x^5 - 5x^4 + 3x^3$, $x^3 + 3x^2 - x - 3$.

In each problem from 13 to 22, find an L.C.M. of the given expressions.

13. 6, 8, 12. **14.** 8, 45, 54.

15. xy, $6xz$, $8yz$. **16.** $4x^2$, $5x^4$, $20x$.

17. $4x^2y^4z^5$, $9x^6y^2z^3$, $6x^4yz^2$. **18.** $2a + 4$, $a - 3$, $a^2 - 9$.

19. $x - 2$, $x + 2$, $x^2 - 4$. **20.** $2x + 8$, $3x - 6$, $x^2 + 2x - 8$.

21. $(x^2 - 49)(x^3 - 8)$, $(x - 7)(x + 7)(x - 2)(x^2 - 4)$, $(x - 3)(x - 2)$.

22. $2x^4 - 2y^4$, $6x^2 + 12xy + 6y^2$, $9x^3 + 9y^3$.

1–24. REDUCTION OF FRACTIONS

From (1–24) under operations with fractions, it follows that a fraction a/b, in which $b \neq 0$, is not changed if both the numerator and the denominator are multiplied or divided by the same quantity, provided that the quantity is not zero. That is, if $k \neq 0$,

(1–49) $$\frac{a}{b} = \frac{ka}{kb},$$

or

(1–50) $$\frac{a}{b} = \frac{a/k}{b/k}.$$

For example,
$$\frac{1}{2} = \frac{2 \cdot 1}{2 \cdot 2} = \frac{2}{4}, \text{ and } \frac{4}{6} = \frac{4/2}{6/2} = \frac{2}{3}.$$

The fundamental principles of (1–49) and (1–50) are applied in reducing a fraction to lowest terms and in changing two or more fractions with different denominators into equivalent fractions with a common denominator.

The *reduction of a fraction to lowest terms,* that is, to a form in which all common non-zero factors are removed from both the numerator and the denominator, is accomplished as follows:

First, factor both the numerator and the denominator into prime factors.

Then, divide both the numerator and the denominator by all their common factors.

It should be noted that this reduction can also be accomplished by dividing the numerator and the denominator by their highest common factor.

The following examples will illustrate the reduction of fractions to lowest terms.

Example 1–29. Reduce $\frac{30}{42}$ to lowest terms.

Solution: Factoring the numerator and the denominator and dividing both by their common factors, we have

$$\frac{30}{42} = \frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 7} = \frac{5}{7},$$

The common factors of the numerator and the denominator are 2 and 3.

Example 1–30. Reduce $\frac{6xy^2}{2x^2y}$ to lowest terms.

Solution: Factoring into prime factors and dividing out common factors, we have

$$\frac{2 \cdot 3 \cdot x \cdot y \cdot y}{2 \cdot x \cdot x \cdot y} = \frac{3y}{x}.$$

In this fraction the common factors are 2, x, and y.

Example 1–31. Reduce $\frac{6x^3 - 24xy^2}{6x^2 - 3xy - 18y^2}$ to lowest terms.

Solution: We have

$$\frac{6x^3 - 24xy^2}{6x^2 - 3xy - 18y^2} = \frac{2 \cdot 3x(x - 2y)(x + 2y)}{3(x - 2y)(2x + 3y)} = \frac{2x(x + 2y)}{2x + 3y}.$$

It is important to note that in a fraction of the type $\frac{a + y}{a + x}$, where the numerator and the denominator have a common term, any attempt to simplify the fraction by cancelling out this common term can lead only to an absurdity. For, quite obviously, $\frac{a + y}{a + x}$ does not equal either $\frac{1 + y}{1 + x}$ or $\frac{y}{x}$ unless $a = 1$ or 0. For example, if we attempt one of these simplifications with the fraction $\frac{2 + 3}{2 + 4}$, we reach the obvious contradiction $\frac{5}{6} = \frac{4}{5}$ or $\frac{5}{6} = \frac{3}{4}$. To avoid this common error, it is important to remember that only common factors, not common terms, may be cancelled. One should make certain that the common quantity is a factor of the entire numerator and of the entire denominator.

1–25. SIGNS ASSOCIATED WITH FRACTIONS

It follows from (1–49) that we can multiply both the numerator and the denominator of a fraction by -1 without changing the

value of the fraction. However, if just one of these is multiplied by -1, then the sign of the fraction must be changed in order to keep its value unaltered. Thus, in effect, a minus sign before a fraction can be moved to either the numerator or the denominator without altering the value of the fraction. We have, as previously stated in (1–22),

$$-\frac{n}{d} = \frac{-n}{d} = \frac{n}{-d},$$

and

$$-\frac{1}{x-y} = \frac{-1}{x-y} = \frac{1}{-(x-y)} = \frac{1}{y-x}.$$

We see, then, that any two of the three signs associated with a fraction, namely, the signs of the numerator, the denominator, and the fraction, can be changed without changing the value of the fraction. In general, the rules for changing signs in a fraction are as follows:

Changing the signs in an even number of factors in the numerator or in the denominator, or in both, does not change the sign of the fraction.

Changing the signs in an odd number of factors in the numerator or in the denominator, or in both, does change the sign of the fraction.

Example 1–32. Find the missing quantities in $\frac{2x}{5} = -\frac{-2x}{(\ )} = \frac{(\ )}{-5}$.

Solution: $\frac{2x}{5} = -\frac{-2x}{5} = \frac{-2x}{-5}$.

Example 1–33. Change the fraction $\frac{a}{y-x}$ to an equivalent fraction with denominator $x - y$.

Solution: Since $x - y = -(y - x)$, we make the following changes in signs:

$$\frac{a}{y-x} = \frac{-a}{-(y-x)} = \frac{-a}{x-y},$$

or

$$\frac{a}{y-x} = -\frac{a}{-(y-x)} = -\frac{a}{x-y}.$$

EXERCISE 1–7

1. Find the missing quantity in each of the following equalities:

a. $\frac{4}{7} = \frac{(\ )}{21}$. **b.** $\frac{(\ )}{15} = \frac{3x}{5}$. **c.** $\frac{-a}{2} = \frac{(\ )}{-8a}$.

d. $-\frac{-3}{6x^2} = \frac{x^2}{(\ )}$. **e.** $\frac{2x^2}{3y} = \frac{12x^3y^2}{(\ )}$. **f.** $\frac{x-a}{2x+a} = \frac{(\ )}{4ax+2a^2}$.

g. $\frac{2x}{1-x^2} = \frac{-2x}{(1+x)(\ )}$. **h.** $\frac{x+y}{x-y} = \frac{(\ )}{y-x}$. **i.** $\frac{x-a}{x-b} = \frac{(\quad)}{b^2-x^2}$.

2. Reduce each of the following fractions to lowest terms.

a. $\dfrac{102010}{350470}$.
b. $\dfrac{8a^2b^3}{12a^4b}$.
c. $\dfrac{6x^4y^8}{20x^2y^7}$.

d. $\dfrac{36x^4y^5z^3}{96x^3y^7z^2}$.
e. $\dfrac{48x^5y^8}{24x^4y^6 + 40x^4y^5}$.
f. $\dfrac{x^4 - 6x^2}{3x^3 - 18x}$.

g. $\dfrac{a + 2b}{4a^2 + 8ab}$.
h. $\dfrac{x^2y^3 - x^3y^2}{x - y}$.
i. $\dfrac{9x^2 - 49}{3x^2 + 13x + 14}$.

j. $\dfrac{2x^2 - 7x - 15}{x^2 - 25}$.
k. $\dfrac{6x^2 - 5x - 6}{4x^2 + 10x - 24}$.
l. $\dfrac{x^4 - 81}{x^6 + 729}$.

m. $\dfrac{x^2 - (y - 3)^2}{4 - (x - y)^2}$.
n. $\dfrac{a^3 - 8b^3}{a^2 - 4b^2}$.
o. $\dfrac{x^3 - 27y^3}{x^2 + 3xy + 9y^2}$.

p. $\dfrac{x^2 - (a + b)x + ab}{[(2x + a)^2 - (x + 2a)^2](x^2 - b)}$.
q. $\dfrac{(x + y)(x - y)(y - z)}{(y^2 - x^2)(z - y)}$.

r. $\dfrac{18x^2 + 3xy - 10y^2}{21x^2 - 26xy + 8y^2}$.
s. $\dfrac{9a^2 + 6ab + b^2}{6a^2 - ab - b^2}$.

t. $\dfrac{x^2 + xy}{x^3 + x^2y + xy^2 + y^3}$.
u. $\dfrac{x^2 + 2xy}{x^2 + x + 2y - 4y^2}$.

1–26. ADDITION AND SUBTRACTION OF FRACTIONS

The sum of two fractions with the same denominator is the fraction whose numerator is the sum of the numerators and whose denominator is the given common denominator. That is, in (1-25),

(1–25)
$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$

For example,
$$\frac{x}{3} + \frac{2y}{3} = \frac{x + 2y}{3}.$$

To add fractions with different denominators, first change the fractions to equivalent fractions having a common denominator, and then write the sum of the new numerators over this common denominator.

Ordinarily, the common denominator which is chosen is a least common multiple of the given denominators, since this leaves the fewest possible common factors in the numerator and denominator of the resultant fractions. However, the same result is obtained, after concellation of all common factors, no matter what common denominator is used. The method of finding an L.C.M. was developed in Section 1–23.

The difference of two fractions, $\frac{a}{b} - \frac{c}{d}$, has been defined in Section 1–2. Thus, by (1–10), when neither b nor d is zero,

(1–51)
$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right).$$

Applying (1–22), we have

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}. \tag{1–52}$$

The following procedure is suggested for adding (or subtracting) fractions.

1. Find a least common multiple of the given denominators.
2. Change each fraction to an equivalent fraction having the L.C.M. as the denominator in the following way: For each fraction, note which factors are in the L.C.M. but not in the denominator of the given fraction. These factors may be found by dividing the L.C.M. by the denominator of the given fraction. Then multiply the numerator and the denominator of the given fraction by these factors.
3. Write the sum (or difference) of the numerators of the new fractions found in step 2 over the L.C.M., and reduce the resulting fraction to lowest terms.

The following examples will indicate a procedure which should be followed until some skill in working problems has been attained.

Example 1–35. Express $\frac{3}{4} - \frac{5}{6} + \frac{7}{9}$ as a single fraction reduced to lowest terms.

Solution: Step 1. The methods in Section 1–23 give us 36 as the L.C.M. of the denominators.

Step 2. Divide the L.C.M. successively by the denominators 4, 6, and 9 to get

$$36/4 = 9,\ 36/6 = 6, \text{ and } 36/9 = 4.$$

Change the given fractions to fractions having 36 as the denominator in the following way:

$$\frac{3 \cdot 9}{4 \cdot 9} = \frac{27}{36},\quad \frac{5 \cdot 6}{6 \cdot 6} = \frac{30}{36},\quad \frac{7 \cdot 4}{9 \cdot 4} = \frac{28}{36}.$$

Step 3. Combine the new fractions to obtain

$$\frac{3}{4} - \frac{5}{6} + \frac{7}{9} = \frac{27}{36} - \frac{30}{36} + \frac{28}{36} = \frac{27 - 30 + 28}{36} = \frac{25}{36}.$$

Note that adding (or subtracting) the numerators and the denominators of the given fractions leads to absurdities. Thus, it is *not* true that $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$. Also, dropping the common denominator leads to absurdities. Thus, it is *not* true that $\frac{1}{2} + \frac{1}{3} = 5$.

Example 1–36. Express $\frac{3x}{x+2} - \frac{2}{2-x} - \frac{5x-2}{x^2-4}$ as a single simplified fraction.

Solution: Step 1. Since $x^2 - 4 = (x+2)(x-2)$ and $-\frac{2}{2-x} = \frac{2}{x-2}$, an

L.C.M. of the denominators is $x^2 - 4$. The given expression may be written as follows:

$$\frac{3x}{x+2} + \frac{2}{x-2} - \frac{5x-2}{x^2-4}.$$

Step 2. When the L.C.M. is divided by the denominators $(x + 2)$, $(x - 2)$, and $(x^2 - 4)$, the results are

$$\frac{x^2-4}{x+2} = x - 2, \quad \frac{x^2-4}{x-2} = x + 2, \quad \frac{x^2-4}{x^2-4} = 1.$$

Multiplying the numerators and the denominators of the given fractions by these factors, we obtain

$$\frac{3x\cdot(x-2)}{(x+2)\cdot(x-2)} = \frac{3x^2-6x}{x^2-4},$$
$$\frac{2\cdot(x+2)}{(x-2)\cdot(x+2)} = \frac{2x+4}{x^2-4},$$
$$\frac{5x-2}{x^2-4} = \frac{5x-2}{x^2-4}.$$

Step 3. Therefore, the desired result is obtained in the following way:

$$\frac{3x}{x+2} + \frac{2}{x-2} - \frac{5x-2}{x^2-4} = \frac{(3x^2-6x) + (2x+4) - (5x-2)}{x^2-4}$$
$$= \frac{3x^2-9x+6}{x^2-4} = \frac{3(x-1)(x-2)}{x^2-4} = \frac{3(x-1)}{x+2}.$$

EXERCISE 1–8

Perform each of the indicated operations and express the answer in lowest terms.

1. $\frac{1}{2} - \frac{2}{3} + \frac{7}{8}$.
2. $1 - \frac{5}{9} - \frac{7}{9}$.
3. $\frac{3}{7} - 5 + \frac{2}{21}$.
4. $\frac{7}{8} + \frac{5}{6} - \frac{5}{12}$.
5. $-3 + \frac{13}{20} - \frac{5}{10}$.
6. $\frac{5}{2} - \frac{17}{5} + \frac{19}{4}$.
7. $\frac{1}{1+x} - \frac{1}{x-1}$.
8. $\frac{2}{1-x} - \frac{1}{1+x} + \frac{3}{1-x^2}$.
9. $\frac{3x-2}{x^3-1} + \frac{x-4}{x-1}$.
10. $\frac{2x}{x+2} - \frac{3x^2}{x-1} + x$.
11. $\frac{1}{x} + \frac{3}{x^2} - \frac{5}{x^3}$.
12. $\frac{1}{x-1} + 2 - \frac{1}{1-x}$.
13. $\frac{1}{x-1} - \frac{2}{x^2-1} + \frac{3}{(x-1)^2}$.
14. $\frac{x-2}{x+2} - \frac{x+2}{2-x} - \frac{8x}{4-x^2}$.
15. $\frac{x-2}{x^2+5x+6} + \frac{x+2}{x^2+7x+12} + \frac{x+3}{x^2+6x+8}$.
16. $\frac{3}{x^2-3x} - \frac{2}{x-3} + \frac{1}{x}$.
17. $1 + x + \frac{2x^2}{1-x}$.
18. $\frac{x}{x+1} + \frac{x+1}{x} - \frac{2x^2-3}{x^2+x}$.
19. $\frac{2x}{x^2-y^2} - \frac{3x}{x^2-2xy+y^2}$.

20. $\dfrac{3}{4 - x^2} - \dfrac{1}{x - 2}$.

21. $\dfrac{a + b}{a - b} + \dfrac{a - b}{a + b} - \dfrac{2ab}{a^2 - b^2}$.

22. $\dfrac{4x}{x - 1} + 4 + \dfrac{6x}{2 - x}$.

23. $\dfrac{1}{x} + \dfrac{2}{y} + \dfrac{3}{xy}$.

24. $\dfrac{x}{x - y} - \dfrac{y}{x - y} + \dfrac{xy}{x^2 - y^2}$.

25. $\dfrac{x}{x + y} + \dfrac{y}{x - y} - \dfrac{xy}{x^2 - y^2}$.

26. $\dfrac{3x}{x^2 - y^2} - \dfrac{2x}{x^2 - 2xy + y^2} - \dfrac{4xy}{x^3 - x^2y - xy^2 + y^3}$.

1–27. MULTIPLICATION AND DIVISION OF FRACTIONS

The product of two fractions is the fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators of the given fractions. That is, by (1–29), if neither b nor d is 0,

(1–29)
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

To illustrate,

$$\frac{2}{3} \cdot \frac{6}{7} = \frac{2 \cdot 6}{3 \cdot 7} = \frac{4}{7}, \quad \text{and} \quad \frac{x - 1}{x + 2} \cdot \frac{x + 3}{x - 2} = \frac{x^2 + 2x - 3}{x^2 - 4}.$$

A special case of (1–29) is worth noting. To multiply a fraction by a number or expression, we multiply the numerator by that number or expression. Thus,

$$a \cdot \frac{b}{c} = \frac{a}{1} \cdot \frac{b}{c} = \frac{a \cdot b}{1 \cdot c} = \frac{ab}{c}.$$

For example,

$$2 \cdot \frac{3}{5} = \frac{6}{5}, \quad \text{and} \quad (x - 1) \cdot \frac{y}{x} = \frac{(x - 1)y}{x}.$$

The quotient of two fractions, $\dfrac{a}{b} \Big/ \dfrac{c}{d}$, has been defined in Section 1–2 and evaluated in Section 1–7. Thus, if no factor in a denominator is zero,

(1–30)
$$\frac{a}{b} \Big/ \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

To illustrate,

$$\frac{5}{7} \Big/ \frac{3}{2} = \frac{5}{7} \cdot \frac{2}{3} = \frac{5 \cdot 2}{7 \cdot 3} = \frac{10}{21}, \text{ and } \frac{3x^2}{y^3} \Big/ \frac{6x^5}{y^7} = \frac{3x^2}{y^3} \cdot \frac{y^7}{6x^5} = \frac{3x^2 \cdot y^7}{y^3 \cdot 6x^5} = \frac{y^4}{2x^3}.$$

In practice, we divide out all factors common to the numerator and denominator before proceeding with the actual multiplication. If necessary, the numerator and the denominator of each given fraction should be factored.

Example 1–37. Multiply $\dfrac{x^2 - 2x}{2x^2 + 5x + 3}$ by $\dfrac{2x^2 - 3x - 9}{x^2 - 9}$ and simplify the result.

Solution: The work may be indicated as follows:

$$\frac{x^2 - 2x}{2x^2 + 5x + 3} \cdot \frac{2x^2 - 3x - 9}{x^2 - 9} = \frac{x(x-2)(x-3)(2x+3)}{(x+1)(2x+3)(x+3)(x-3)}$$
$$= \frac{x(x-2)}{(x+1)(x+3)} = \frac{x^2 - 2x}{x^2 + 4x + 3}.$$

Example 1–38. Divide $\dfrac{x^2 - 5x + 4}{x^2 - 3x - 4}$ by $\dfrac{x^3 - 4x^2 + x - 4}{2x - 1}$.

Solution: By (1–30),

$$\frac{x^2 - 5x + 4}{x^2 - 3x - 4} \div \frac{x^3 - 4x^2 + x - 4}{2x - 1} = \frac{(x^2 - 5x + 4) \cdot (2x - 1)}{(x^2 - 3x - 4) \cdot (x^3 - 4x^2 + x - 4)}$$
$$= \frac{(x-1)(x-4)(2x-1)}{(x+1)(x-4)(x-4)(x^2+1)}$$
$$= \frac{(x-1)(2x-1)}{(x+1)(x-4)(x^2+1)}.$$

1–28. COMPLEX FRACTIONS

The fractions we have been discussing so far may be called *simple fractions,* to distinguish them from the fractions which we now discuss.

A fraction which contains other fractions in the numerator or in the denominator is called a *complex fraction.* Since the simplification of a complex fraction is essentially a problem in division, we first reduce the numerator and the denominator of the complex fraction to simple fractions and then proceed as in division.

Example 1–39. Simplify $\dfrac{1 + \frac{y}{x}}{\frac{1}{x} + \frac{1}{y}}$.

First Solution: The numerator of the given complex fraction reduces to the simple fraction $\dfrac{x + y}{x}$, and the denominator reduces to $\dfrac{y + x}{xy}$. Hence, we have

$$\frac{\frac{x+y}{x}}{\frac{y+x}{xy}} = \frac{(x+y) \cdot xy}{x \cdot (y+x)} = y.$$

Alternate Solution: Frequently it may be more convenient to multiply both the numerator and the denominator of the complex fraction by an L.C.M. of the denominators of all simple fractions occurring in the given complex fraction. In this example, the simple fractions are $\dfrac{y}{x}$, $\dfrac{1}{x}$, and $\dfrac{1}{y}$, and an L.C.M. of their denominators is xy.

Therefore, by multiplying the numerator and the denominator of the complex fraction by xy, we get

$$\frac{\left(1 + \frac{y}{x}\right) \cdot xy}{\left(\frac{1}{x} + \frac{1}{y}\right) \cdot xy} = \frac{xy + y^2}{y + x} = \frac{(x+y)y}{y+x} = y.$$

EXERCISE 1–9

In each of the problems from 1 to 20, perform the indicated operations and express the answer as a simple fraction in lowest terms.

1. $\frac{3}{5} \cdot \frac{18}{155}$.

2. $\frac{2}{3} \cdot \frac{5}{7}$.

3. $\frac{3}{7} \cdot \frac{x}{6y}$.

4. $\frac{4y}{9x} \cdot \frac{6x^2}{16y^3}$.

5. $\frac{14}{9} \Big/ \frac{28}{45}$.

6. $\frac{3x}{17y^3} \Big/ \frac{16x^3z}{68y^6w}$.

7. $\frac{6a^2b^3}{15a^3b^5} \cdot \frac{3a^3b}{48ab^2}$.

8. $\frac{27x^3y}{72xy^3} \Big/ \frac{15y^2}{16x}$.

9. $\frac{15a^4b^4}{15a^3} \Big/ \frac{54ab^2}{14a^3b^2}$.

10. $\frac{3x^2}{4y} \cdot \frac{10z}{27x^3} \cdot \frac{8y^4}{15z^3}$.

11. $\frac{9x^4y^3}{x^2 - 25} \cdot \frac{x^2 - 6x + 5}{6x^3y^2}$.

12. $\frac{x^3 - 8}{9 - x^2} \cdot \frac{x^2 + 2x - 3}{x^2 + 2x + 4}$.

13. $\frac{x^2 + 2x - 3}{(x - 7)^2} \div \frac{x^2 + x - 6}{x^2 - 5x - 14}$.

14. $\frac{x^4 + 27x}{x^4 - 4x^2} \div \frac{x^3 + 3x^2 - 2x - 6}{x(x^2 - 5x + 6)}$.

15. $\frac{27x^3 - 8}{6x^2 + 19x + 10} \cdot \frac{4x^2 - 25}{9x^2 - 12x + 4}$.

16. $\frac{6x^2 + 5x - 21}{4x^2 - 9x - 28} \cdot \frac{6x^4 + 36x^3}{9 - 4x^2} \cdot \frac{2x^2 - 5x - 12}{9x^3 + 21x^2}$.

17. $\frac{2x^2 + 5x + 3}{5x^2 - 24x - 5} \cdot \frac{3x^2 - 20x + 12}{x^2 + 3x + 2} \div \frac{6x^2 + 5x - 6}{4x^2 + 9x + 2}$.

18. $\frac{x^2 - (2x - 3z)^2}{(x - 3z)^2 - 4y^2} \div \left[\frac{4x^2 - (3z - x)^2}{(2y - x)^2 - 9z^2} \cdot \frac{9z^2 - (x - 2y)^2}{(3z - 2y)^2 - x^2}\right]$.

19. $\left[\frac{8x^2 - 2x - 15}{4y^6z^9} \div \frac{9 - 4x^2}{36y^7z^7}\right] \cdot \frac{4x^2 + 12x + 9}{48x^2y^3 + 60y^3}$.

20. $\frac{(x^4 - 625)(x^2 - 9)}{(x + 5)^2(x - 5)^3} \div \frac{3x^4 + 75x^2}{x^2 + 10x + 25}$.

In each of the problems from 21 to 38, simplify the complex fraction.

21. $\dfrac{\frac{2}{3} \cdot \frac{3}{4} \cdot \frac{7}{2}}{\frac{5}{16}}$.

22. $\dfrac{\frac{5}{9} \cdot \frac{3}{10} \cdot \frac{18}{19}}{\frac{64}{81}}$.

23. $\dfrac{\frac{1}{2} - \frac{2}{3} + \frac{3}{4}}{\frac{1}{5} + \frac{1}{6} + \frac{1}{8}}$.

24. $\dfrac{\frac{7}{18} - \frac{4}{15} + \frac{3}{10}}{\frac{1}{2} - \frac{3}{5} + \frac{1}{6}}$.

25. $\dfrac{\frac{6xy}{5}}{\frac{4x^2}{2y}}$.

26. $\dfrac{\frac{16x^2y^3}{9x}}{\frac{18x^3y}{4xy}}$.

27. $\dfrac{8x^3 - y^3}{\frac{2x - y}{y^2}}$.

28. $\dfrac{x + \frac{3}{x} - 3}{\frac{1}{6} + \frac{7}{6x}}$.

29. $\dfrac{a + \frac{14}{a} - 7}{\frac{a}{3} - \frac{7}{3a} + \frac{1}{3}}$.

30. $\dfrac{\frac{1}{x^2} - \frac{4}{x} + 7}{\frac{2}{x^2} - \frac{3}{x} - 4}$.

31. $\dfrac{\frac{1}{x^4} - \frac{4}{x^2} + 4}{\frac{1}{x^2} - 4}$.

32. $\dfrac{3 - \frac{2}{x + 2}}{9 + \frac{20}{x^2 - 4}}$.

33. $\dfrac{x - 5 + \dfrac{8}{x+4}}{x - 3 - \dfrac{8}{x+4}}$.
34. $\dfrac{x - 2 - \dfrac{1}{x-2}}{x - 4 - \dfrac{1}{x-4}}$.
35. $\dfrac{\dfrac{x}{2x+1} + \dfrac{1-2x}{x}}{\dfrac{x}{2x+1} - \dfrac{1-2x}{x}}$.

36. $\dfrac{\dfrac{1-x}{1+x} - \dfrac{1+x}{1-x}}{\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1}}$.
37. $\dfrac{\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}}{\dfrac{x^2+y^2}{x^2-y^2} - \dfrac{x^2-y^2}{x^2+y^2}}$.
38. $\dfrac{\dfrac{x-2y}{x+1}\left[1 - \dfrac{2y}{2y-x}\right]}{1 - \dfrac{x+4y^2}{x^2-1} + \dfrac{1}{x-1}}$.

Simplify each of the following expressions:

39. $\dfrac{\dfrac{x + \frac{1}{x}}{1 - \frac{1}{x}} + \dfrac{x - \frac{1}{x}}{1 + \frac{1}{x}}}{x + \dfrac{1}{x-1}}$.
40. $\dfrac{x + \dfrac{1}{x}}{x - 1 + \dfrac{1}{x + \dfrac{1}{x+1}}}$.
41. $\dfrac{x + \dfrac{1}{x}}{1 - \dfrac{1}{1 + \dfrac{1}{x-1}}}$.

42. $\dfrac{\dfrac{x^2+y^2}{y} - 1}{\dfrac{1}{x} - \dfrac{1}{y}} \div \dfrac{x^3+y^3}{x^2-y^2}$.
43. $\dfrac{\dfrac{1}{x^3} - \dfrac{1}{y^3}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}} - \dfrac{\dfrac{1}{x^2} - \dfrac{1}{y^2}}{\dfrac{1}{x} + \dfrac{1}{y}}$.

44. $\left(\dfrac{x+8}{x-1} - x\right)\left(\dfrac{x}{7x-4} - \dfrac{1}{x+2}\right) \div \left(x - 6 + \dfrac{4}{x-2}\right)$.

45. $\left(4y - \dfrac{x^2}{x-y}\right)\left(y - \dfrac{xy - x^2 - y^2}{x-2y}\right) \div \left(2 - \dfrac{3x}{x+y}\right)$.

1–29. LINEAR EQUATIONS

Introduction. In Section 1–13, an equation was defined as a statement of equality between two algebraic expressions. In this section we shall discuss equations in one unknown of the simplest type, called *linear* equations, typified by the form

$$ax = b, \tag{1–53}$$

where a and b are specified numbers and $a \neq 0$. Some ideas pertaining to equations in general are needed first.

Solution or Root of an Equation. We shall use the term *unknown* to designate a literal quantity that appears in an equation and is not regarded as specified at the outset. A system of values of the unknowns which, when substituted for them, makes the equation a true assertion is called a *solution* of the equation. A solution is also called a *root* when only one unknown is involved. Thus, the values $x = 2$ and $y = -1$ define a solution of the equation $3x + 2y = 4$, since the equation is *satisfied* for these values; that is, $3(2) + 2(-1) = 4$.

Equivalent Equations. Two equations are *equivalent* if they have exactly the same solutions. For example, $2x + 11 = 7x - 4$ and $4x + 22 = 14x - 8$ are equivalent, since, as will be seen, each has exactly one root, namely, $x = 3$. However, $2x + 11 = 7x - 4$ and $2x^2 + 11x = 7x^2 - 4x$ are *not* equivalent, because the latter equation has, in addition to the root $x = 3$, the root $x = 0$.

Operations on Equations. Each of the following operations on an equation yields an equivalent equation:

Adding the same expression to (or subtracting the same expression from) both sides.

Multiplying (or dividing) both sides by the same non-zero number.

For any solution of $F = G$ also makes $F + H = G + H$ true, and is therefore a solution of this latter equation. Conversely, any solution of $F + H = G + H$ is one of $F = G$, because

$$F = F + H - H = G + H - H = G.$$

Likewise, any solution of $F = G$ is one of $aF = aG$, provided that a is a non-zero number, and conversely.

Transposition of Terms. Transposing a term of an equation consists in moving the term from one side of the equation to the other and changing its sign. This operation is equivalent to adding the same quantity to both sides (or subtracting the same quantity from both sides). For example, consider the equation

$$2x + 11 = 7x - 4.$$

If we *transpose* $2x$ from the left side to the right, and transpose -4 from the right to the left, we obtain

$$11 + 4 = 7x - 2x.$$

In effect, we subtracted $2x$ from each side and added 4 to each side.

1–30. LINEAR EQUATIONS IN ONE UNKNOWN

An equation in the form shown in (1–53) is called a *linear equation in one unknown.* We shall show that such a linear equation has one and only one root, namely,

$$x = \frac{b}{a}. \tag{1–54}$$

If each side of the equation $ax = b$ is divided by a, the result is

$$x = \frac{b}{a},$$

which is equivalent to the given equation. Therefore, (1–53) clearly has one and only one solution, namely, b/a.

It should be noted that if we allow a to be 0 in (1–53), there are two possibilities. Either no solution exists, when $b \neq 0$, since for no number x is it true that $a \cdot x = 0 \cdot x = 0 = b \neq 0$; or else every number x is a solution, when $b = 0$, since $a \cdot x = 0 \cdot x = 0 = b$ for all values of x.

An equation which is not apparently linear may frequently be solved by the theory of linear equations, by replacing the given equation by a linear equation to which it is equivalent. The following steps serve as a guide to the method to be used when there is only one unknown in the equation.

1. Clear the equation of any fractions with numerical denominators by multiplying both sides of the equation by a least common multiple of the denominators of those fractions.

2. Transpose all terms containing the unknown to one side and all other terms to the other side. We may collect the terms containing the unknown on either side.

3. Combine like terms. If the equation now assumes the form in (1–53), it is a linear equation and can be solved by dividing both sides by the coefficient of the unknown.

4. Check the result obtained in step 3 by substituting in the original equation. While it is desirable to include step 4 to show that the number x found in step 3 is actually a solution of the equation, step 4 is not a necessary part of the solution process, since the operations performed in the preceding steps always yield equivalent equations. The purpose of step 4 is to help to make certain that there has been no error.

In step 2, we generally transpose terms containing the unknown to whichever side makes solving the equation easier.

The following explanations should be noted carefully.

Students at times "transpose" coefficients of the unknown. Thus, $2x = 6$ takes the erroneous form $x = 6 - 2$ by "transposing" 2 to the right side. The correct procedure is to remove the coefficient 2 by *division,* since it is a multiplier of x. Therefore, we should divide both sides of the equation $2x = 6$ by 2 and have

$$\frac{2x}{2} = \frac{6}{2},$$

or

$$x = 3.$$

Errors of this type may be avoided if the student applies the rules of algebra properly and checks his solutions carefully.

Multiplying or dividing both sides of an equation by a polynomial involving the unknown will not necessarily yield an equivalent equation. When the operation is multiplication, the new equation thus obtained may have roots in addition to the roots of the original equation. These extra roots are called *extraneous roots.* We then say that the equation is *redundant* with respect to the original equation. When both sides of an equation are divided by a polynomial involving the unknown, the new equation may lack some of the original roots. It is then said to be *defective* with respect to the original equation.

If both sides of an equation are multiplied by a polynomial involving the unknown, the check in step 4 of the recommended procedure is a necessary step in the solution process. An extraneous root can thus be identified. Dividing both sides of an equation by a polynomial involving the unknown is not a permissible procedure, since roots that are lost cannot be regained.

It should be noted that a non-linear equation may sometimes be treated so as to obtain a linear equation which is possibly redundant. For example, the fractions in the equations of Problems 39 to 48 of Exercise 1–10 may be eliminated by multiplying both sides of each equation by a least common multiple of the denominators of the fractions in that equation. Since this multiplier contains the unknown, the new equation may have solutions which are not solutions of the given equation. Hence, the solutions must be checked to see whether or not they actually are solutions of the given equation.

Example 1–40. Solve the equation $\frac{2x}{3} = 4 - \frac{x+7}{5}$.

Solution: We clear the equation of fractions by multiplying both sides by 15, which is an L.C.M. of the denominators of the fractions, and obtain

$$10x = 60 - 3x - 21.$$

Collecting the terms containing x on the left side and all other terms on the right, we have

$$10x + 3x = 60 - 21.$$

Combining like terms, we obtain

$$13x = 39.$$

Finally, dividing both sides by 13, we have

$$x = 3.$$

To check, substitute 3 for x in the original equation. The result is

$$\frac{2 \cdot 3}{3} = 4 - \frac{3+7}{5}, \quad \text{or} \quad 2 = 4 - 2, \quad \text{or} \quad 2 = 2.$$

Therefore, 3 is the root.

Example 1–41. Solve the equation $6x - 3y - 1 = 5y + 2x + 11$ for y in terms of x. In this case, regard x as specified.

Solution: Collect the terms in the unknown y, on the right side, and collect all other terms on the left. The result is

$$6x - 2x - 1 - 11 = 5y + 3y.$$

Combining like terms, we have

$$4x - 12 = 8y.$$

Now we divide both sides by the coefficient of the unknown to obtain

$$\frac{4x - 12}{8} = y.$$

Then,

$$y = \frac{4x - 12}{8} = \frac{x - 3}{2}.$$

To check, substitute $\frac{x - 3}{2}$ for y in the original equation, and obtain

$$6x - \frac{3(x - 3)}{2} - 1 = \frac{5(x - 3)}{2} + 2x + 11.$$

This equation reduces to

$$12x - 3(x - 3) - 2 = 5(x - 3) + 4x + 22,$$

which becomes

$$12x - 3x + 9 - 2 = 5x - 15 + 4x + 22,$$

or

$$9x + 7 = 9x + 7.$$

Since this result is an identity, $y = \frac{x - 3}{2}$ is the solution of the original equation, regardless of the value of x.

Example 1–42. Find two consecutive integers such that four times the first is equal to six times the second diminished by 20.

Solution: Let x be the smaller integer, and $x + 1$ the next larger integer. Then, from the statement of the problem, we have

$$4x = 6(x + 1) - 20.$$

From this equation, we obtain

$$4x = 6x + 6 - 20.$$

Then,

$$14 = 2x, \quad \text{or} \quad x = 7.$$

Hence $x = 7$ and $x + 1 = 8$ are the two consecutive integers. The student should carefully check these values by substitution in the original statement of the problem.

Example 1–43. The speed of an airplane in still air is 400 miles per hour. If it requires 20 minutes longer to fly from A to B against a wind of 50 miles per hour than it does to fly from B to A with the wind, what is the distance from A to B?

Solution: Let x be the distance from A to B. From the data, the speed of the plane against the wind, in miles per hour, is $400 - 50 = 350$, and the speed of the plane with the wind, in miles per hour, is $400 + 50 = 450$.

Since distance = rate · time, or $d = rt$, we have $\frac{d}{r} = t$. Hence,

$$\frac{x}{350} = \text{time, in hours, required to fly from } A \text{ to } B,$$

and

$$\frac{x}{450} = \text{time, in hours, required to fly from } B \text{ to } A.$$

Therefore, from the statement of the problem, it follows that

$$\frac{x}{350} - \frac{x}{450} = \frac{1}{3}.$$

Solving this equation, we have

$$9x - 7x = 1050, \quad \text{or} \quad x = 525.$$

So the distance from A to B is 525 miles.

EXERCISE 1–10

Solve for the unknown in each problem from 1 to 15.

1. $-4x - 2 = 3 - 2x$.
2. $3x - 6 = x + 12$.
3. $3x - 2 = -4x - 5$.
4. $3y + 7 = 2 - 2y$.
5. $-9x - 7 = 6 - \frac{5x}{2}$.
6. $4w + \frac{5}{2} = 3w - \frac{1}{2}$.
7. $4 - 3z = 6(1 + 2z)$.
8. $4x - 6 = 3x + 4$.
9. $x - 8 = 2x + 3$.
10. $6y + 7 = 5y + 6$.
11. $10x - 3 = 9x + 4$.
12. $\frac{3}{4}x - \frac{5}{2} = \frac{3}{2} - \frac{1}{4}x$.
13. $5x - 7 - 8x = 4x - 17 - 6x$.
14. $10y + 4 + 6y = 2y + 7 + 3y$.
15. $6(5 + 4x) - 3(x - 4) = 0$.

Solve for y in terms of x in each problem from 16 to 24.

16. $2x - y = 3$.
17. $23x + 2y = 4$.
18. $x - 4y + 10 = 0$.
19. $4y - 2x + 2 = 0$.
20. $\frac{x}{a} + \frac{y}{b} = 1$.
21. $6x - 2y = 3$.
22. $3(x - 4) + 4y = -3$.
23. $2x + 4(y - 3) = 5$.
24. $3x + 7\left(y - \frac{1}{3}\right) = \frac{4}{3}$.

Solve for all values of x which satisfy the equation in each problem from 25 to 48.

25. $5x - 3 = 4(x - 2)$.
26. $8(x - 2) - 9(x - 4) = 13$.
27. $\frac{4x + 5}{3} = 11$.
28. $\frac{2x + 17}{3} = -7$.
29. $\frac{9 - 4x}{6} = \frac{4}{3}$.
30. $\frac{5 + 6x}{3} - 4 = 0$.
31. $\frac{x + 5}{4} = \frac{x + 7}{5}$.
32. $\frac{x - 6}{5} = \frac{x - 8}{7}$.
33. $\frac{x + 3}{5} = \frac{2x - 7}{12}$.
34. $\frac{2x - 1}{3} = \frac{x - 5}{7}$.
35. $\frac{2x + 3}{4} - \frac{3x - 1}{2} = 3$.
36. $\frac{7x + 3}{6} = \frac{2}{5}x + \frac{1}{2}$.
37. $\frac{8x - 21}{5} + \frac{1}{4} = \frac{5x}{6}$.
38. $\frac{7 - x}{4} - \frac{4x + 3}{7} = \frac{6x}{5} - \frac{16}{3}$.
39. $3 - \frac{4}{x} = 6\left(-\frac{4}{3}\right)$.
40. $\frac{4}{9 - 2x} + 3 = 7$.

41. $\dfrac{2}{x} + \dfrac{3}{x^2} = \dfrac{4}{x} - \dfrac{5}{x^2}$.

42. $\dfrac{1}{x} - \dfrac{1}{9} = \dfrac{1}{9} - \dfrac{1}{x}$.

43. $\dfrac{4}{2x + 3} + \dfrac{5}{x - 4} = \dfrac{1}{4x^2 - 10x - 24}$.

44. $\dfrac{1}{x} + \dfrac{3}{x - 1} = \dfrac{4}{x + 1}$.

45. $\dfrac{3}{6x^2 - 2x + 1} - \dfrac{1}{2x^2 - 4x + 7} = 0$.

46. $\dfrac{x - 1}{x + 2} - \dfrac{19 - 22x}{x^2 - x - 6} = \dfrac{x + 1}{x - 3}$.

47. $\dfrac{5}{6(x - 3)} + \dfrac{7 - x}{2(x^2 + 3x + 9)} = \dfrac{5 - 2x - x^2}{81 - 3x^3}$.

48. $\dfrac{4}{x} + \dfrac{4 + 4x}{x^2 + 4x} = \dfrac{-5}{x + 4}$.

49. Divide 98 into two parts such that one of them exceeds the other by 18.

50. Find three consecutive integers whose sum is 84.

51. Find two consecutive integers whose squares differ by 13.

52. If 8 times a certain number is 9 more than 5 times the same number, what is the number?

53. A rectangular plot of ground is four times as long as it is wide. If its perimeter is 4,800 feet, what is its area?

54. How many pounds of coffee at 90 cents per pound and how many pounds at 98 cents per pound will it take to make 100 pounds of a mixture costing 96 cents per pound?

55. At a college play, admission was 25 cents for a child and 75 cents for an adult. If \$210 was taken in from 500 admissions, how many children and how many adults were admitted?

56. A man has \$365 in 41 bills of \$5 and \$10 denominations. How many bills of each denomination does he have?

57. What are the angles of a triangle, if one angle is three times the second angle and six times the third angle?

58. One man, X, can do a certain job in 7 days, and another man, Y, can do the same job in 15 days. How long would it take them to do the job working together?

2 The Function Concept

2–1. RECTANGULAR COORDINATE SYSTEMS IN A PLANE

In Section 1–5 we saw how we can associate a real number with every point on a number scale. The real number attached to a given point is called the *coordinate* of the point. This representation suggests the assumption that to any real number there corresponds precisely one point on the scale, and to any point of the scale there corresponds precisely one real number. This *one-to-one correspondence* between the set of real numbers and points on the number scale is known as a *one-dimensional coordinate system.*

We shall now extend the concept of a one-dimensional coordinate system to a system of coordinates in a plane in which two number scales are perpendicular to each other. The two perpendicular lines, which we shall call *coordinate axes,* divide the plane into four parts, or *quadrants,* numbered as shown in Fig. 2–1. The horizontal and vertical lines are designated as the *x-axis* and the *y-axis,* respectively, and their point of intersection is called the *origin* and is labeled O.

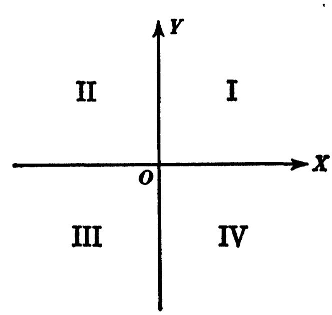

FIG. 2–1.

On each of these axes, we construct a number scale by selecting an arbitrary unit of length and the origin as the zero point. As in Section 1–5, a coordinate on the x-axis will be considered positive if it is to the right of O, that is, to the right of the y-axis, and will be negative if it is to the left. A coordinate on the y-axis will be considered positive if it is above the x-axis, and negative if it is below.

Just as the real-number scale of Fig. 1–1 gave us a system of one-dimensional coordinates by which we could set up a one-to-one correspondence between points on a line and real numbers, so the

system of coordinates with respect to two mutually perpendicular axes sets up a one-to-one correspondence between points in a plane and *ordered number pairs*. We use the designation "ordered" pairs for the following reason. To designate any point, we shall agree to give its directed distance from the y-axis first, and call it the *abscissa* or *x-coordinate,* and then the directed distance from the x-axis and call it the *ordinate* or *y-coordinate.* The abscissa and ordinate of a point constitute its *rectangular coordinates.* They are written in parentheses as an ordered number pair, as in the notation (x, y), the abscissa always being written first. By this scheme we assign to each point of the plane a definite ordered pair (x, y) of real numbers and, conversely, to each ordered pair (x, y) of real numbers there is assigned a definite point of the plane.

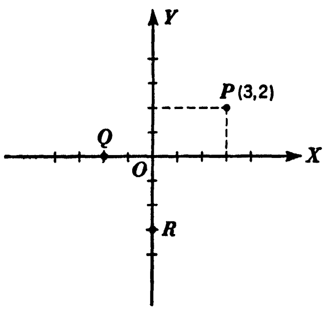

FIG. 2–2.

Thus, the abscissa of the point P in Fig. 2–2 is 3, and its ordinate is 2, and we say that the coordinates of the point P are $(3, 2)$. Similarly, the coordinates of Q are $(-2, 0)$, the coordinates of R are $(0, -3)$, and those of the origin are $(0, 0)$. Marking in the plane the position of a point designated by its coordinates is called *plotting* the point.

The coordinate system we have constructed is a particular case of *cartesian coordinates,* so called in honor of Rene Descartes (1596-1650), who first introduced a coordinate system in 1637. It is called a *rectangular* system, since the axes intersect in a right angle. (Actually the axes may intersect at any angle, but it is usually simpler to take them perpendicular to each other. When the two axes are not perpendicular, the coordinate system is called an *oblique coordinate system.* Oblique systems will not be used in this book.)

2–2. DISTANCE BETWEEN TWO POINTS

It was seen in Section 1–10 that $|a - b|$ equals the distance between two points on the number scale represented by the real numbers a and b. It follows that $|x_2 - x_1|$ represents the distance between the points $A(x_1, 0)$ and $B(x_2, 0)$ on the x-axis of Fig. 2–3.

Let us now consider the two points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_1). The points have the same y-coordinate, which means that they lie on the same horizontal line. Hence, the distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_1)$ is the same as

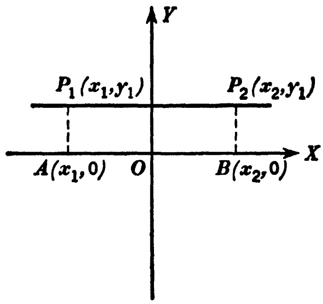

FIG. 2–3.

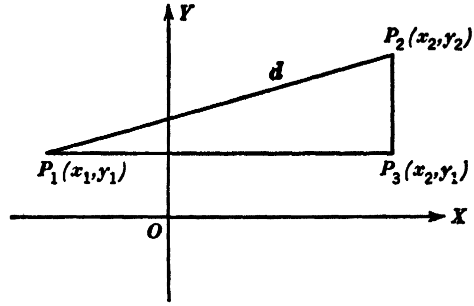

FIG. 2–4.

the distance between $A(x_1, 0)$ and $B(x_2, 0)$. This distance is $|x_2 - x_1|$. Similarly, we can show that $|y_2 - y_1|$ represents the distance between two points (x_2, y_1) and (x_2, y_2) on a vertical line. We have thus arrived at the following two important properties.

If two points $P_1(x_1, y_1)$ and $P_2(x_2, y_1)$ have the same y-coordinate, then the distance between them, or $|P_1P_2|$, is given by

$$|P_1P_2| = |x_2 - x_1|. \tag{2–1}$$

If two points $Q_1(x_1, y_1)$ and $Q_2(x_1, y_2)$ have the same x-coordinate then the distance between them is given by

$$|Q_1Q_2| = |y_2 - y_1|. \tag{2–2}$$

The concept of the distance between any two points in a plane is so important that we shall now develop a formula for it. Let us denote by d the distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. That is, d is the length of the line segment P_1P_2 in Fig. 2–4. Let P_3 be the point (x_2, y_1) as shown.

Since the angle at P_3 is a right angle, we have, by the Pythagorean theorem,

$$|P_1P_2|^2 = |P_1P_3|^2 + |P_3P_2|^2.$$

Therefore,

$$\begin{aligned} |P_1P_2|^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2. \end{aligned}$$

That is, the distance d between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{2–3}$$

This formula is known as the *distance formula.*

Example 2–1. Find the distance between the points $(-3, 2)$ and $(5, 2)$.

Solution: It makes no difference which point is labeled P_1. Let us label the first one P_1 and the second one P_2. Since the two points have the same y-coordinate, (2–1) applies and $|P_1P_2| = |5 - (-3)| = |5 + 3| = |8| = 8$.

Example 2–2. Find the distance between the points (3, 1) and (3, 7).

Solution: Again let us label the first point P_1 and the second one P_2. Since the points have the same x-coordinate, (2–2) applies and $|P_1P_2| = |7 - 1| = 6$.

Example 2–3. Find the distance between the points (2, − 1) and (5, 3).

Solution: Let us designate the points as $P_1(2, -1)$ and $P_2(5, 3)$. By (2–3), the distance $|P_1P_2|$ is

$$d = \sqrt{(5 - 2)^2 + (3 - (-1))^2}$$
$$= \sqrt{9 + 16} = \sqrt{25} = 5.$$

We shall now consider a special application of the distance formula (2–3). The line segment OP from the origin O to a point $P(x, y)$ is called the *radius vector* to P. By (2–3), the distance between $O(0, 0)$ and any point $P(x, y)$, or the length of the radius vector to P, is

$$d = \sqrt{(x - 0)^2 + (y - 0)^2}$$

or

(2–4) $$d = \sqrt{x^2 + y^2}.$$

Thus, for the point $P(3, 2)$, the radius vector has the length $\sqrt{3^2 + 2^2} = \sqrt{13}$.

EXERCISE 2–1

1. Plot the following points:

a. (3, 5). **b.** (− 4, 7). **c.** (5, − 2).
d. (− 3, − 6). **e.** (1, − 3). **f.** (− 8, − 6)
g. (0, 2). **h.** (− 5, 0). **i.** (3, 0).

2. In each of the following cases, plot the pair of points and find the distance between them:

a. (2, 3) and (7, 3). **b.** (5, − 2) and (− 1, − 2).
c. (1, 4) and (1, 0). **d.** (− 3, − 2) and (− 3, 4).
e. (2, 1) and (5, 6). **f.** (0, − 8) and (5, − 3).
g. (3, 2) and (5, 7). **h.** (− 4, − 6) and (− 8, − 6).

3. Find the length of the radius vector to each of the following points:

a. (4, 3). **b.** (12, 5). **c.** (1, − 1).
d. (5, − 12). **e.** (7, 0). **f.** (− 3, − 2).
g. (0, 4). **h.** (− 1, 2). **i.** (a, b).
j. $(1, \sqrt{3})$. **k.** $(-\sqrt{3}, -1)$. **l.** (m, n).

2–3. FUNCTIONS

So far we have been concerned with single numbers and pairs of numbers. Now we shall consider mathematical relations, known

as functions, between two sets of numbers. To distinguish between the two sets, we shall call one of these sets the *domain of definition* X of the function, and the other the *range set* Y of the function.

We begin by defining a *variable* to be a symbol which may take any value in a given set of numbers. If x is a symbol which is used to denote any number of the domain X and y is a symbol which denotes any number of the set Y, then x is called an *independent variable* and y is called a *dependent variable.* If a set contains only a single number, the symbol used to represent that number is called a *constant.*

To set forth a function, the domain should be explicitly specified; that is, it is necessary to determine definitely just what elements or numbers the domain contains. The same is true of the range set. Then, as soon as a definite rule of correspondence is given which assigns to each number x of the domain one or more numbers y of the range set, the function is specified completely. We thus have a set of ordered pairs of numbers (x, y), where x is any number of X and y is a number of Y.

The set of ordered pairs of numbers (x,y) is called the function. The rule of correspondence which determines the collection of pairs (x,y) is often expressed by a formula involving algebraic or other processes. In such cases we usually find it convenient to refer to the formula as though it were the function. For example, we often speak of the function $y = x^2 - 3x + 5$ when actually we mean the set of ordered pairs (x,y) determined by $y = x^2 - 3x + 5$. If just one number of Y is paired with each value of x, the function is said to be *single-valued.* If more than one number of Y is paired with some value of x, the function is said to be *multiple-valued.* We frequently find it possible to deal with multiple-valued functions by separating them into distinct single-valued functions.

The *range* of values of the function consists of those numbers y in the range set Y which actually correspond to some number x of the domain. When the range of a function has been determined, it is always possible to replace the original range set, which may include numbers in addition to those of the range, by the range itself. We shall now consider several examples of functions.

Illustration 1. The *constant function* $y = c$ associates the same number c with every number x of the domain X. Hence, the range Y of the function consists of just one number c. Since y has the same value for all pairs (x, y), the function is evidently single-valued.

Illustration 2. The *identity function* $y = x$ associates with every real number x the number itself. In other words, the numbers

corresponding to $x = 1, 2, 3, \cdots$ are $y = 1, 2, 3, \cdots$ respectively. The domain X is the set of all real numbers, and the range Y is also this entire set. The function $y = x$ is single-valued, because it associates just one number of Y with each value of x.

Illustration 3. Let us consider the *linear function* defined by the equation $y = 3x - 2$. The domain X is the set of all real numbers, and the range Y is also this entire set. In this case a given x determines a unique y which is equal to $3x - 2$. For example, corresponding to $x = 1, 2, 3$, we have $y = 1, 4, 7$. The function is single-valued.

Illustration 4. Let the rule of correspondence be given by the equation $y = x^2$. Also, let X be the set of all real numbers, and let Y be the set of all non-negative real numbers. Then, to the number $x = -2$ there corresponds the number $y = (-2)^2 = 4$; to the number $x = 3$ there corresponds the number $y = 9$; and so on. Hence, $y = x^2$ associates just one number of Y with a given value of x, and defines y as a single-valued function of x. Although this is not obvious, the range of the function is the given range set Y.

Illustration 5. In this case let the rule of correspondence be given by the equation $y^2 = x$. Here X is the set of all non-negative real numbers, and Y is the set of all real numbers. Then to the number $x = 2$ there correspond the two numbers $y = \sqrt{2}$ and $y = -\sqrt{2}$; to the number $x = 9$ there correspond the numbers $y = \sqrt{9} = 3$ and $y = -\sqrt{9} = -3$; and so on. Thus, $y^2 = x$ defines y as a two-valued function of x. Only for $x = 0$ is there a single value of y, namely, $y = 0$.

Illustration 6. The function $y = \sqrt{a^2 - x^2}$, with domain X consisting of all real numbers x such that $-a \leqq x \leqq a$, is a single-valued function with range set Y consisting of the numbers $0 \leqq y \leqq a$. Here the range is Y, as may be proved.

Note that $\sqrt{a^2 - x^2}$ has no meaning when $a^2 - x^2$ is negative. Hence, those values of x must be excluded for which $a^2 - x^2 < 0$, and we must restrict the value of x to the interval $-a \leqq x \leqq a$. (We shall consider the meaning of the square root of a negative number in Chapter 11.) The function $y = \sqrt{a^2 - x^2}$ is single-valued because $\sqrt{a^2 - x^2}$ represents the non-negative square root of $a^2 - x^2$. (The other square root of $a^2 - x^2$ is denoted by $-\sqrt{a^2 - x^2}$.)

Illustration 7. The function $y = 1/x$ is defined for all real numbers different from 0. For $x = 0$, $1/x$ is not defined, since division by zero is not permissible. In other words, although there are

values of y for values of x near 0, there is no possible value for y when x actually equals 0.

Usually the functions which we are about to consider are defined for all values of x, with the following two exceptions:

Values of x must be excluded which involve even roots of negative numbers, since these are not defined as real numbers.

Values of x must be excluded for which a denominator is zero, since division by zero is not a permissible operation.

On occasion, the use to be made of a function will restrict the values of x for which it is to be regarded as defined.

2–4. FUNCTIONAL NOTATION

Since functions are mathematical entities, they may be given letter notations, such as f, g, ϕ. To designate the number, or numbers, y corresponding to a given number x according to the rule specified by a given function f, we use the notation $f(x)$. As an illustration, let the function f be defined by the equation

$$y = x^2 - 2x + 3.$$

Then $f(0) = 3$, $f(-1) = 6$, and so on. Frequently, the symbol $f(x)$ is used to designate the function rather than the functional values. The context will make the meaning clear.

It should be remembered that the notation $y = f(x)$ does not mean that y is a number f multiplied by another number x. Instead it is an abbreviation for "f of x."

The set X does not have to be as simple as in the preceding illustrations. If X should consist of a set of ordered pairs of numbers, the rule of correspondence would then determine a value, or values, of y for each ordered pair of X. We would then have a function of *two independent variables*. For example, the area of a triangle is given by the relationship

$$A = f(a, b) = \frac{1}{2}ab.$$

Here X is the set of ordered pairs, (a, b), of positive real numbers, where a is the length of the altitude of the triangle and b is the length of its base; and Y is the set of numbers A, each of which represents an area corresponding to a given pair (a, b). Similarly, a function of three variables, $f(x, y, z)$, is defined in terms of a set X of ordered triples (x, y, z). Thus, if $f(x, y) = x^2 + y^2$, then $f(2, 3) = 2^2 + 3^2 = 13$. Also, if $f(x, y, z) = x - y + 2z$, then $f(3, 2, 5) = 3 - 2 + 2(5) = 11$.

EXERCISE 2-2

By using the phrase "a function of" in each problem from 1 to 8, express each given quantity, which is regarded as a dependent variable, as a function of one or more independent variables. Where possible write the relationship both in words and in symbols.

1. The area of a circle.
2. The area of a triangle.
3. The area of a trapezoid.
4. The volume of a sphere.
5. The volume of a cylinder.
6. The retail price of food in a grocery store.
7. The annual premium for a life insurance policy.
8. A person's height.
9. Given $f(x) = 2x - 3$, find $f(0)$, $f(-1)$, $f(3)$, $f(1/2)$, $f(\sqrt{2})$, $3f(1)$, $f(3)/4$, $f(y)$, $f\left(\frac{1}{x}\right)$, $\frac{1}{f(x)}$.
10. Given $g(x) = 3x^2 + 5$, find $g(1)$, $g(-3)$, $(g(2))^2$, $g(4)$, $2g(4)$, $g(z - 1)$.
11. Using $f(x)$ and $g(x)$ as defined in problems 9 and 10, find $\frac{f(-2)}{g(3)}$, $f(6)g(3)$, $f(g(y))$.

In problems 12 to 26, let $f(x, y) = 2xy + 3x - 2y$, and let $g(a, b, c) = a^2 + b^2 + c^2$. Evaluate or simplify each given expression.

12. $f(1, 2)$.
13. $f(0, 0)$.
14. $g(0, 0, 0)$.
15. $g(1, 2, 3)$.
16. $f(x, 1/x)$.
17. $g(p, q, r)$.
18. $f(x, y) + g(x, y, z)$.
19. $\frac{f(0, 0)}{g(0, 1, 2)}$.
20. $\frac{f(-1, 0)}{g(0, 0, 1)}$.
21. $f(1, -1) \cdot g(\sqrt{2}, \sqrt{3}, \sqrt{5})$.
22. $\frac{f(y, z)}{g(y, x, z)}$.
23. $\frac{f(1, -1)}{g(3, 2, 1)} - \frac{f(-1, 1)}{g(-3, -2, -1)}$.
24. $g(-a, -b, -c)$.
25. $g(-a, b, c) - g(a, -b, c)$.
26. $g(a, b, -c)$.
27. Express the area A and circumference of C of a circle as functions of the radius r. By eliminating r, express A as a function of C, and also express C as a function of A.
28. Express the volume of a sphere as a function of its surface area.
29. Express the surface area of a sphere as a function of its volume.
30. Suppose that U is a function of V and that V is a function of W. Show that U is a function of W.

Determine the maximal domain of values of x for which y is defined as a function of x in each problem from 31 to 61. Assume that x and y are real numbers.

31. $y = x$.
32. $y = 3x$.
33. $y = -\frac{x}{2}$.
34. $y = 3x + 1$.
35. $y = 2x - 3$.
36. $y = 4x + 5$.
37. $y = x^2$.
38. $y = x^2 - 2$.
39. $y = x^2 + 1$.
40. $y = x(2x + 1)$.
41. $y = (3x - 1)(x + 1)$.
42. $y = (2x - 3)(2x + 1)$.
43. $y = x(x - 1)(x + 1)$.
44. $y = x^2 + \frac{1}{x}$.
45. $y = \frac{1}{x(x - 1)}$.
46. $x^2 + y^2 = 9$.
47. $x^2 - y^2 = 9$.
48. $x^2 - y^2 = 0$.
49. $x^2 + y^2 = 0$.
50. $3x^2 + y^2 = 0$.
51. $y = \frac{x^2}{1 + x^2}$.

52. $y = \dfrac{1}{1 - x}\cdot$ **53.** $y = \dfrac{1}{x + 2} - \dfrac{1}{x}\cdot$ **54.** $y = \dfrac{x^2}{x + 1}\cdot$

55. $y = \dfrac{x^2 - 1}{x^2 + 1}\cdot$ **56.** $x = 4y^2.$ **57.** $x + 1 = 3y^2.$

58. $y = \sqrt{1 - x^2}.$ **59.** $y = \sqrt{4 - x^2}.$ **60.** $y = \sqrt{x^2 + 1}.$

2–5. SOME SPECIAL FUNCTIONS

The following additional illustrations of two rather unusual, but very useful, functions are given here to help us become better acquainted with the function concept.

The *absolute-value function* is defined by associating with x its absolute value $|x|$. The functional relation is given by $y = |x|$. Thus, the domain comprises the set of all real numbers, while the range comprises the set of all non-negative real numbers. For this function we have the values 2, 3, 0, π, $\sqrt{2}$ corresponding, respectively, to $x = 2, -3, 0, \pi, -\sqrt{2}$.

The *bracket function* or *greatest-integer function,* represented by the notation $y = [x]$, is defined as the largest integer which does not exceed x. Its domain is the set of all real numbers, and its range is the set of all integers. Thus, if $f(x) = [x]$, then $f(-3.5) = -4$, $f(-1) = -1$, $f(0) = 0$, $f(2.5) = 2$, and $f(5) = 5$.

EXERCISE 2–3

1. Given $f(x) = |x|$, find $f(-3)$, $f(2.3)$, and $f(f(x))$.
2. Given $f(x) = [x]$, find $f(3.2)$, $f(2)$, and $f(-3)$.
3. Given $f(x) = x - [x]$, find $f(-3)$ and $f(2.5)$.
4. Given $f(x) = |x| + [x] - x$, find $f(-3.5)$ and $f(4)$.
5. Given $f(x) = [x] + [2 - x] - 1$, find $f(0)$, $f(-1/2)$, $f(1)$, $f(3/2)$, and $f(2)$.
6. Let $f(x)$ be the function whose domain X is the set of all real numbers for which the definition is as follows:

$$\text{if } x < 0, \text{ then } f(x) = -x;$$
$$\text{if } x \geqq 0, \text{ then } f(x) = x.$$

Find $f(3)$ and $f(-2)$.

2–6. VARIATION

A particularly important example of a simple type of function often occurring in the physical sciences is given by the formula

$$y = kx.$$

If $k > 0$, this equation shows that *y increases as x increases, and that y decreases as x decreases.* We usually say that *y varies directly*

as x, or that y *is directly proportional to* x. If $x \neq 0$, the relationship may also be written as follows:

$$\frac{y}{x} = k,$$

where k is called the *constant of proportionality*. This relationship is equivalent to saying that the ratio of y to x remains constant for all non-zero values of x.

The value of the constant k in any particular problem may be determined from a known pair of values of x and y in the problem. Thus, if the given relationship is $y = kx$, and if we know that $y = 6$ when $x = 2$, then $k = 3$. The formula then becomes $y = 3x$.

We say that y *varies inversely as* x, or y *is inversely proportional to* x, if

$$y = \frac{k}{x} \qquad (x \neq 0).$$

This relationship shows that y *decreases as* x *increases, and that* y *increases as* x *decreases*. But, when $x \neq 0$, the following two formulas are equivalent:

$$xy = k \text{ and } y = \frac{k}{x}.$$

Therefore, the relationship between x and y is such that the product of x and y is constant.

Several types of variations may be combined in a single equation to express a certain law. For example, when y varies directly as x and z, we say that y *varies jointly with* x *and* z, and we write

$$y = kxz.$$

Direct variation and inverse variation are often combined in applications. Thus, according to Newton's law of gravitation, the force F of attraction between two bodies of masses m_1 and m_2 varies directly as the product of their masses and inversely as the square of the distance d between them. The equation is

$$F = \frac{km_1m_2}{d^2}.$$

Example 2–4. If a man is paid \$15 for an 8-hour day, how much would he make in a 35-hour week?

Solution: The wages a man earns vary directly as the amount of time he works. Since this is a problem in direct variation, we have

$$w = kt.$$

In this formula w represents the total wages, in dollars; t represents the time worked, in hours; and the constant k represents the wage rate in dollars per hour. Substituting $w = 15$ and $t = 8$ determines the constant $k = 1.875$. The general formula then becomes

$$w = 1.875\, t.$$

Therefore, when $t = 35$, we have

$$w = (1.875)(35) = 65.62.$$

Hence, the man earns \$65.62 in a week.

Example 2–5. A motorist traveling at an average rate of 50 miles per hour made a trip in 5 hours. How long would it take him to make the same trip at an average rate of 60 miles per hour?

Solution: Since the time required varies inversely as the speed, we have

$$t = \frac{k}{r}.$$

In this case, k represents the distance traveled, in miles; r is the speed, in miles per hour; and t is the time required, in hours. Substituting $t = 5$ and $r = 50$ determines $k = 250$. Hence, the general formula is

$$t = \frac{250}{r}.$$

Therefore, when $r = 60$, we obtain

$$t = \frac{250}{60} = \frac{25}{6}.$$

Hence, the time required is 4 hours 10 minutes.

Example 2–6. Under suitable conditions the electric current I in a conductor varies directly as the electromotive force E and inversely as the resistance R of the conductor. When $E = 110$ volts and $R = 10$ ohms, $I = 11$ amperes. Find what voltage is necessary to cause 2 amperes to flow through 60 ohms of resistance.

Solution: From the statement of the problem, we see that the combined variation is given by the formula

$$I = \frac{kE}{R}.$$

Substituting $I = 11$, $E = 110$, and $R = 10$ determines the constant k to be 1. Therefore, the formula becomes

$$I = \frac{E}{R}.$$

This relationship may also be expressed as follows: The required voltage E is equal to the current I flowing through the conductor multiplied by the resistance R, or $E = IR$.

For the specified values, $E = (2)(60) = 120$. Hence, $E = 120$ volts.

In this problem $k = 1$. The formula obtained is commonly known as Ohm's law. It is widely applied to entire and partial circuits through which electric currents flow.

EXERCISE 2–4

1. If y varies directly with x, and $y = 15$ when $x = 7$, find a formula for y in terms of x.
2. If x varies directly with y, and $x = 32$ when $y = 4$, find x when $y = 3$.
3. If y is directly proportional to x^2, and $y = 112$ when $x = 4$, find y when $x = 9$.
4. If y is inversely proportional to x, and $y = -2$ when $x = 1$, find y when $x = -3$.

5. If y varies inversely with x, and $y = 10$ when $x = 3$, find y when $x = 6$.

6. If x varies directly with y and inversely with z, and $x = 4$ when $y = 12$ and $z = 2$, find x when $y = 16$ and $z = 4$.

7. If y varies directly with $\sqrt{x}$ and inversely with z^2, and $y = 18$ when $x = 9$ and $z = 2$, find y when $x = 25$ and $z = 6$.

8. If y is directly proportional to x and inversely proportional to $\sqrt{z}$, and $y = 4$ when $x = 1$ and $z = 1$, find y when $x = 2$ and $z = 4$.

9. If y varies directly with x^3 and inversely with $1 - z^2$, and $y = 2$ when $x = 1$ and $z = 2$, find y when $x = -1$ and $z = -2$.

10. If two spheres have radii r_1 and r_2, diameters d_1 and d_2, and surface areas S_1 and S_2, respectively, show that
$$\frac{r_1^2}{r_2^2} = \frac{d_1^2}{d_2^2} = \frac{S_1}{S_2}.$$

11. If the two spheres in problem 10 have volumes V_1 and V_2, respectively, show that
$$\frac{V_1}{V_2} = \frac{r_1^3}{r_2^3} = \frac{d_1^3}{d_2^3}.$$

12. If the radii of two spheres are 3 units and 1 unit, respectively, find *a*) the ratio of their surface areas and *b*) the ratio of their volumes.

13. What are the ratio of the surface areas and the ratio of the volumes of two spheres if the ratio of their radii is 3/2?

14. The diameter of the planet Jupiter is approximately 10.9 times the diameter of Earth. Assuming that both planets are spheres, find *a*) the ratio of their surface areas and *b*) the ratio of their volumes.

15. The diameter of the sun is approximately 109 times the diameter of Earth. Compare the volumes and the surface areas of the sun and Jupiter, if we assume that the sun and both planets are spheres.

16. By how much must the diameter of a sphere be multiplied to give a sphere whose surface area is 25 times that of a given sphere?

17. When the volume V of a gas remains constant, the pressure P varies directly as its absolute temperature T. (Absolute temperature is measured from the so-called absolute zero, which is approximately $-460°$ F or $-273°$ C.) If gas is enclosed in a tank having a volume of 1,000 cubic feet and the pressure is 54 pounds per square inch at a temperature of 27° C, what will be the temperature when the pressure is raised to 108 pounds per square inch?

18. If the temperature T of a gas remains constant, the pressure P varies inversely as the volume V. A gas at a pressure of 50 pounds per square inch has a volume of 1,000 cubic feet. If the pressure is increased to 150 pounds per square inch while the temperature remains constant, what is the volume?

19. The weight of a body above the earth's surface varies inversely as the square of the distance from the center of the earth. If a certain body weighs 100 pounds when it is 4,000 miles from the center of the earth, how much will it weigh when it is 4,010 miles from the center and when 4,100 miles from the center?

20. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. A copper wire 10 inches long and 0.04 inches in diameter has a resistance of 0.0656 ohms, approximately. What is the resistance of a copper wire 1 inch long and 0.01 inches in diameter?
21. What is the diameter of a copper wire 1,000 inches long whose resistance is 10 ohms?
22. According to Kepler's third law, the square of the time it takes a planet to make one circuit about the sun varies as the cube of its mean distance from the sun. The mean distance of the earth is 92.9 million miles, and the mean distance of Jupiter is 475.5 million miles. Find the time it takes Jupiter to make one circuit about the sun.

2–7. CLASSIFICATION OF FUNCTIONS

It is often desirable to group functions into classes. For our immediate purpose it will suffice to consider a classification into *algebraic* functions and non-algebraic, or *transcendental,* functions. Let us first give a more precise definition of a polynomial function and then define algebraic functions and give some illustrations of both.

A *polynomial function* of x is a function given by the relationship

$$y = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real constants, $a_0 \neq 0$, and n is a positive integer or zero. The polynomial function is said to be of *degree n.* The function which makes the number 0 correspond to every number x is also called the *zero polynomial,* but this polynomial has no degree.

A *rational function* of x is a function which either is a polynomial function or can be expressed as a quotient of two polynomials. Thus, a polynomial is often referred to as a *rational integral function of x.*

A *polynomial, or a rational integral function, of x, y, z,* $\cdots$, is defined to be the algebraic sum of terms of the form

$$kx^ay^bz^c \cdots,$$

where k is a constant coefficient and each of the exponents a, b, c, $\cdots$ is either a positive integer or zero*. The degree of such a function is the degree of the highest-degree term which is present.

For example, the expressions $3x^2 - 5$ and $5x^2 - 7xy^2 + 6z$ define polynomial functions of the second degree and third degree, respectively. These and the expressions $\dfrac{x-y}{x+y}$ and $2x - \sqrt{7} + \dfrac{x}{x^2+1}$ are examples of rational functions.

* Zero exponents will be defined in Chapter 4. For now, one needs only note that $u^0 = 1$ for any non-zero number u.

A number is an *algebraic number* if it is a root of a polynomial equation of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

in which the coefficients $a_0, a_1, \cdots a_n$ are integers, not all zero.

Analogous to the term algebraic number, we have the term *algebraic function*. A function $y = f(x)$ is called an *algebraic function of x* if y is a solution of an equation of the form

$$P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_{n-1}(x)y + P_n(x) = 0,$$

where the coefficients $P_0(x), P_1(x), \cdots, P_n(x)$ are polynomials in x, and n is a positive integer.

Polynomials and rational functions are special types of algebraic functions. The functions that we have considered so far were illustrations of algebraic functions. According to our definition, $y = \sqrt{x}$ is an algebraic function of x because $y^2 - x = 0$. In this case, $n = 2$, $P_0(x) = 1$, $P_1(x) = 0$, and $P_2(x) = x$. Similarly, $y = \sqrt{\dfrac{x^2-1}{x}}$ is an algebraic function of x because $xy^2 - x^2 + 1 = 0$. Here $n = 2$, $P_0(x) = x$, $P_1(x) = 0$, and $P_2(x) = -x^2 + 1$.

An *irrational function* is an algebraic function which is not a rational function.

A *transcendental number* is a number which is not algebraic, and a *transcendental function* is a function which is not algebraic.

Functions like the trigonometric functions, which we shall take up in Section 3–2, belong to the class of transcendental functions. Later we shall consider other types of transcendental functions, namely, the logarithmic and exponential functions such as $\log x$ and 10^x.

3 The Trigonometric Functions

3-1. THE POINT FUNCTION P(t)

The trigonometric functions that we are about to define are functions in the sense previously described in Section 2-3; that is, they are relations between two sets of numbers. The student who is familiar with the trigonometric functions from his high-school work is cautioned to note that we are not, for the present, discussing angles in connection with these functions. We shall see that the concept of a trigonometric function need not be associated with an angle; in fact, many of the most important applications of mathematics in modern science and engineering are concerned with trigonometric functions of pure numbers. Hence, we shall adopt the numerical point of view, leaving the study in terms of angles as a secondary consideration.

Consider a circle with a radius of one unit placed at the origin of a rectangular-coordinate system. See Fig. 3-1. Let t be any real

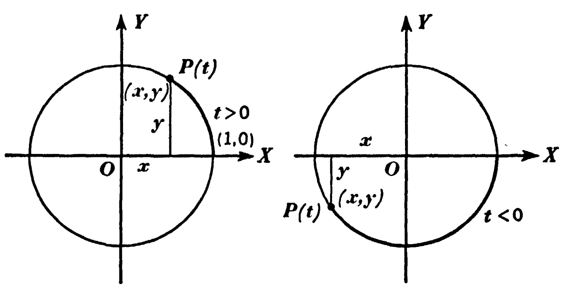

FIG. 3-1.

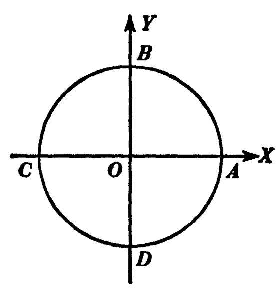

FIG. 3-2.

number. Starting at the point with coordinates $(1, 0)$, we lay off on the unit circle an arc of length $|t|$. If $t > 0$, we measure the arc in a counterclockwise direction. If $t < 0$, we measure in the clockwise direction. If $t = 0$, the arc consists only of the point $(1, 0)$. By this procedure, there is associated with each real number t a definite end-point $P(t)$ of the arc whose initial point is $(1, 0)$. There-

fore, corresponding to every real number t, we have a definite ordered pair (x, y) of numbers which are the coordinates of the endpoint of the arc.

Since $P(t)$ lies on the unit circle, it is at one unit distance from the origin. Hence, it follows from the distance formula that

$$x^2 + y^2 = 1. \tag{3–1}$$

By means of this equation we can find the second coordinate of the point $P(t)$, except for sign, if one of the coordinates is known.

To determine the number pair (x, y) for the point $P(t)$ corresponding to a given value of t, we shall take note of the fact that the circumference of the unit circle is $2\pi = 6.2832$ units (approximately). For example, since arc ABC in Fig. 3–2 is one-half of the complete circumference, it is π units in length, and arc AB is equal to $\pi/2$. It now becomes apparent that $P(0)$ is the initial point $(1, 0)$; $P(\pi)$ is the point $(-1, 0)$; $P(\pi/2)$ is the point $(0, 1)$; and $P(3\pi/2)$ and $P(-\pi/2)$ both represent the point $(0, -1)$.

3–2. DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS

The correspondence between the set of real numbers t and the set of ordered pairs (x, y) leads to definitions of the six common trigonometric functions, namely, the sine, cosine, tangent, cotangent, secant and cosecant.

General Relationships. We shall define the *cosine* of the real number t to be x, and the *sine* of the real number t to be y. Thus, we have

$$x = \cos t, \tag{3–2}$$

and

$$y = \sin t. \tag{3–3}$$

The domain of each of these functions (the sine function and the cosine function) is the set of all real numbers t. Since, however, every point $P(t)$ lies on the unit circle, neither of its coordinates (x, y) can exceed 1 in absolute value. Therefore,

$$|\cos t| \leqq 1 \quad \text{and} \quad |\sin t| \leqq 1. \tag{3–4}$$

In other words, the ranges of cos t and sin t are restricted by the requirements $-1 \leqq \cos t \leqq 1$ and $-1 \leqq \sin t \leqq 1$, respectively, for all values of t. It may be shown that the range of each of these functions is the set of all real numbers u such that $-1 \leqq u \leqq 1$.

The other four functions may be defined in terms of the cosine and sine, as follows:

$$\tan t = \frac{\sin t}{\cos t} \qquad (\cos t \neq 0), \tag{3–5}$$

(3–6) $$\cot t = \frac{\cos t}{\sin t} \qquad (\sin t \neq 0),$$

(3–7) $$\sec t = \frac{1}{\cos t} \qquad (\cos t \neq 0),$$

(3–8) $$\csc t = \frac{1}{\sin t} \qquad (\sin t \neq 0).$$

Since the cosine and sine are defined in terms of the coordinates of the point $P(t)$, it is also possible to express the other functions in terms of these same coordinates. From the definitions of the cosine and the sine given by (3–2) and (3–3) and from the definitions of the other functions given by (3–5), (3–6), (3–7), and (3–8), we have

(3–9) $$\tan t = \frac{\sin t}{\cos t} = \frac{y}{x} \qquad (x \neq 0),$$

(3–10) $$\cot t = \frac{\cos t}{\sin t} = \frac{x}{y} \qquad (y \neq 0),$$

(3–11) $$\sec t = \frac{1}{\cos t} = \frac{1}{x} \qquad (x \neq 0),$$

(3–12) $$\csc t = \frac{1}{\sin t} = \frac{1}{y} \qquad (y \neq 0).$$

We note here that cos t, or x, appears in the denominators of both tan t and sec t. Hence, tan t and sec t are not defined when t is a number for which the x-coordinate of $P(t)$ equals zero. For example, since the x-coordinate of $P(\pi/2)$ or $P(3\pi/2)$ is 0, it follows that tan $\pi/2$, sec $\pi/2$, tan $3\pi/2$, and sec $3\pi/2$ are not defined. Similarly, it can be shown that cot 0, csc 0, cot π, and csc π are not defined. We conclude, therefore, that the domain of each of the functions tan t, cot t, sec t, and csc t is the set of all real numbers for which the denominator is not zero.

It also follows from (3–7) and (3–8) that numerical values of sec t or of csc t can never be less than 1. Hence, the ranges of these two functions are restricted by the requirements

(3–13) $$|\sec t| \geqq 1 \text{ and } |\csc t| \geqq 1.$$

From (3–5) and (3–6) we obtain an idea of the behavior of the tangent and cotangent functions. It may be shown that the range of each of these functions is the set of all real numbers.

The Trigonometric Functions of $\pi/6$, $\pi/4$, and $\pi/3$. The computation of the numerical values of the trigonometric functions in general is beyond the scope of this book. However, we shall use the methods of plane geometry to find sin t, cos t, and tan t for $t = \pi/6$,

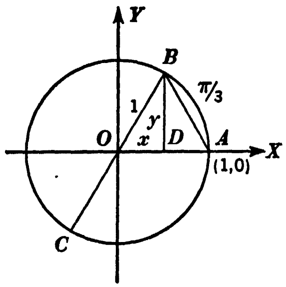

FIG. 3–3.

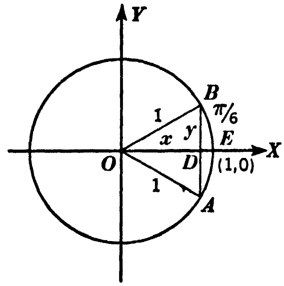

FIG. 3–4.

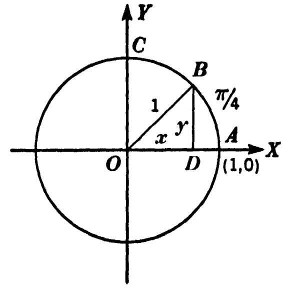

FIG. 3–5.

$\pi/4$, and $\pi/3$, in order to show that for certain values of t the trigonometric functions can be found exactly without tables.

To compute the functions for the real value $t = \pi/3$, we construct the unit circle of Fig. 3–3. Arc AB is given to be equal to $\pi/3$, which is one-sixth of the complete circumference. Triangle OAB is inscribed in the circle, as shown, with side BO extended through the origin to C. Our problem now is to find the values of $x = \cos(\pi/3)$ and $y = \sin(\pi/3)$ as coordinates of the point B.

Since $OA = OB$, triangle AOB is isosceles. Hence,

$$\text{angle } OAB = \text{angle } OBA.$$

We note also that arc $CAB = \pi$ and

$$\text{arc } CA = \text{arc } CAB - \text{arc } AB = \pi - \pi/3 = 2\pi/3.$$

Therefore,

$$\text{arc } CA = 2 \text{ arc } AB.$$

Also,

$$\text{angle } COA = 2 \text{ angle } AOB.$$

Furthermore, angle COA is an exterior angle of triangle AOB. Hence, it equals the sum of the two remote interior angles OAB and OBA, or

$$\text{angle } COA = \text{angle } OAB + \text{angle } OBA.$$

Thus,

$$2 \text{ angle } AOB = \text{angle } OAB + \text{angle } OBA,$$

and triangle AOB is equilateral.

If we draw BD perpendicular to OA, it will bisect OA. Then $x = 1/2$; and from $x^2 + y^2 = 1$ it follows that $y^2 = 3/4$ and $y = \sqrt{3}/2$. Hence, $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, and $\tan(\pi/3) = \sqrt{3}$.

For $t = \pi/6$, place the equilateral triangle AOB of Fig. 3–3 in the unit circle as shown in Fig. 3–4, where E is the mid-point of arc AB. Then arc $EB = \pi/6$ and OE is the perpendicular bisector of chord AB.

Since DB is one-half of chord AB, $y = 1/2$. From $x^2 + y^2 = 1$ it follows that $x^2 = 3/4$ and $x = \sqrt{3}/2$. Hence, $\cos(\pi/6) = \sqrt{3}/2$ and $\sin(\pi/6) = 1/2$. We then have $\tan(\pi/6) = 1/\sqrt{3} = \sqrt{3}/3$.

For $t = \pi/4$, construct a unit circle as shown in Fig. 3–5 with arc $AB = \pi/4$. Since arc $AC = \pi/2$, arc $BC = \pi/2 - \pi/4 = \pi/4$. Hence, arc AB = arc BC, and

$$\text{angle } AOB = \text{angle } BOC.$$

Draw BD perpendicular to OA. Since the two parallels OC and DB are cut by the transversal OB,

$$\text{angle } BOC = \text{angle } OBD.$$

Hence,

$$\text{angle } AOB = \text{angle } OBD,$$

and

$$OD = DB.$$

Thus, $y = x$. Substituting this value of y in $x^2 + y^2 = 1$, we have $2x^2 = 1$ and $x^2 = 1/2$. Therefore, $x = \cos(\pi/4) = \sqrt{2}/2$, and $y = \sin(\pi/4) = \sqrt{2}/2$. It follows that $\tan(\pi/4) = 1$.

Other Special Values. In a similar fashion we can compute exactly the trigonometric functions of such values of t as $\frac{2\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{6}$, and $-\frac{3\pi}{4}$. Functions of multiples of $\pi/2$ may also be computed in this fashion, if one considers a straight line as a right triangle in which one angle is 0 and, hence, one side has zero length.

Example 3–1. Calculate the values of the six trigonometric functions of $t = \pi/2$.

Solution: As explained in Section 3–1, the coordinates of the point $P(\pi/2)$ are (0, 1). Hence, by (3–2), (3–3), (3–9), (3–10), (3–11), and (3–12),

$$\begin{array}{ll} \cos(\pi/2) = 0, & \sec(\pi/2) \text{ is undefined,} \\ \sin(\pi/2) = 1, & \csc(\pi/2) = 1, \\ \tan(\pi/2) \text{ is undefined,} & \cot(\pi/2) = 0. \end{array}$$

EXERCISE 3–1

1. Determine the coordinates of each of the following points:

a. $P(2\pi)$. **b.** $P(-\pi)$. **c.** $P(5\pi/2)$.
d. $P\left(-\frac{7\pi}{2}\right)$. **e.** $P(4\pi)$. **f.** $P(-7\pi)$.

2. In each of the following cases, carefully draw a unit circle and estimate the coordinates of $P(t)$.

a. $P(1)$. **b.** $P(2)$. **c.** $P(3)$.
d. $P(-2)$. **e.** $P(4)$. **f.** $P(-3)$.

3. Evaluate each of the following:

a. $\sin \frac{3\pi}{4}$. b. $\cos \frac{5\pi}{3}$. c. $\tan \left(-\frac{3\pi}{4}\right)$

d. $\cot \frac{2\pi}{3}$. e. $\sin \frac{11\pi}{6}$. f. $\sec \frac{7\pi}{6}$.

g. $\csc \left(-\frac{\pi}{6}\right)$. h. $\sin \left(-\frac{2\pi}{3}\right)$. i. $\cos \left(-\frac{5\pi}{6}\right)$

4. In each of the following cases assume that $0 \leqq t \leqq 2\pi$, and draw a figure showing approximately the appropriate arc (or arcs).

a. $\sin t = 1/2$. b. $\cot t = -1$. c. $\tan t = 1$. d. $\csc t = -1$.

e. $\cos t = -1/2$, $\sin t$ being positive. f. $\cot t = -1$, $\sec t$ being negative.

5. Complete the following table, which shows the algebraic signs of the trigonometric functions in the four quadrants.

Function	Quadrant in which $P(t)$ lies			
	I	II	III	IV
$\cos t$	+	−	−	+
$\sin t$	+	+	−	−
$\tan t$	+	−	+	−
$\cot t$				
$\sec t$				
$\csc t$				

6. Use the equation $x^2 + y^2 = 1$ to find all values of t for which $\tan t = \cot t$, where $0 \leqq t \leqq 2\pi$.

7. Use the equation $x^2 + y^2 = 1$ in each of the following cases to find the other trigonometric functions of t.

a. $\sin t = 1/2$. b. $\cos t = 3/5$. c. $\sec t = 13/12$.

d. $\csc t = -3/2$. e. $\sec t = -2$. f. $\tan t = 4/3$.

g. $\cot t = 2$. h. $\tan t = -6/7$. i. $\sin t = -3/5$.

8. Prove that each of the following equations is correct.

a. $\sin(-t) = -\sin t$. b. $\cos(-t) = \cos t$.

c. $\sec(-t) = \sec t$. d. $\tan(-t) = -\tan t$.

9. For each of the following cases, state the quadrant, or quadrants, in which the given condition is satisfied.

a. The sine and cosine have the same signs.

b. The tangent and cosine have opposite signs.

3–3. IDENTITIES

As an immediate consequence of the definitions of the six trigonometric functions, we can establish certain relationships among

them which hold for every value of t. Since $P(t)$ lies on the unit circle, (3–1) holds; that is, $x^2 + y^2 = 1$. But, according to (3–2) and (3–3), $x = \cos t$ and $y = \sin t$. Therefore, we have the equation

$$\cos^2 t + \sin^2 t = 1. \tag{3–14}$$

This states that "the square of the cosine of t plus the square of the sine of t equals unity." Since (3–14) holds for every value of t, it is an identity. Note that we use the symbol $\cos^2 t$ instead of $(\cos t)^2$. This simplified notation is used for all positive exponents, but is never used in the case of a negative exponent. Thus, $\cos^{-1} t$ does not mean the same as $(\cos t)^{-1}$, as we shall see in Chapter 8.

Similarly, we can prove that for each value of t for which the functions are defined,

$$1 + \tan^2 t = \sec^2 t, \tag{3–15}$$

$$1 + \cot^2 t = \csc^2 t. \tag{3–16}$$

Proof of (3–15): By definition, $\tan t = \dfrac{\sin t}{\cos t}$ and $\sec t = \dfrac{1}{\cos t}\cdot$ However, these relationships have no meaning if $\cos t$ equals zero, that is, if the x-coordinate of $P(t)$ equals zero. When $\cos t \neq 0$, we may divide both sides of the identity $\cos^2 t + \sin^2 t = 1$ by $\cos^2 t$ and obtain

$$1 + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}\cdot$$

Therefore,

$$1 + \tan^2 t = \sec^2 t.$$

Proof of (3–16): By definition, $\cot t = \dfrac{\cos t}{\sin t}$ and $\csc t = \dfrac{1}{\sin t}\cdot$ In this case we assume that $\sin t \neq 0$. When we divide both sides of $\sin^2 t + \cos^2 t = 1$ by $\sin^2 t$, we obtain

$$1 + \frac{\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}\cdot$$

Therefore,

$$1 + \cot^2 t = \csc^2 t.$$

We thus have established the three identities which we restate here for easier reference:

$$\cos^2 t + \sin^2 t = 1, \tag{3–14}$$

$$1 + \tan^2 t = \sec^2 t, \tag{3–15}$$

$$1 + \cot^2 t = \csc^2 t. \tag{3–16}$$

These fundamental identities are very important in working with trigonometric identities and should be remembered.

Our present work with identities will consist of reducing given trigonometric expressions to other forms. Unfortunately, no specific rule of procedure can be given for making these reductions. Profi-

ciency in making such reductions is a matter of both practice and experience. Generally speaking, when we want to reduce a given expression to some other form, it is helpful first to perform any indicated algebraic operations and then to use some form of one of the fundamental relationships to simplify the expression.

To prove an identity, we may proceed in any one of the following ways:

1) We may work on the more complicated member of the identity and attempt to reduce it to the simpler member.

2) We may work with both sides of the identity and show that they reduce to the same expression.

3) We may form the difference of the two sides of the identity and prove that difference to be equal to zero.

It is frequently desirable to express both sides of the given identity in terms of sines and cosines alone, and then use (3–14) if needed.

We shall consider a few examples to illustrate the procedure in reducing expressions.

Example 3–2. Show that $\cos t + \sin t \tan t = \sec t$.

Solution: From (3–5), or the definition of the tangent, we have

$$\cos t + \sin t \tan t = \cos t + \sin t \frac{\sin t}{\cos t}\cdot$$

Adding, we have

$$\cos t + \sin t \frac{\sin t}{\cos t} = \frac{\cos^2 t + \sin^2 t}{\cos t}\cdot$$

Since $\cos^2 t + \sin^2 t = 1$ and $\dfrac{1}{\cos t} = \sec t$,

$$\cos t + \sin t \tan t = \frac{1}{\cos t} = \sec t.$$

Example 3–3. Reduce $\dfrac{1}{\tan t + \cot t}$ to $\sin t \cos t$.

Solution: By definition, $\tan t = \dfrac{\sin t}{\cos t}$ and $\cot t = \dfrac{\cos t}{\sin t}\cdot$ Therefore,

$$\frac{1}{\tan t + \cot t} = \frac{1}{\dfrac{\sin t}{\cos t} + \dfrac{\cos t}{\sin t}} = \frac{1}{\dfrac{\sin^2 t + \cos^2 t}{\cos t \sin t}} = \frac{\cos t \sin t}{\sin^2 t + \cos^2 t}\cdot$$

Since $\sin^2 t + \cos^2 t = 1$, we obtain

$$\frac{1}{\tan t + \cot t} = \frac{\cos t \sin t}{1} = \sin t \cos t.$$

Example 3–4. Establish the identity $(\sec t - \cos t)^2 = \tan^2 t(1 - \cos^2 t)$ by reducing both sides to the same expression.

Solution: By definition, $\sec t = \dfrac{1}{\cos t}\cdot$ Hence, the left side becomes

$$\left(\frac{1}{\cos t} - \cos t\right)^2 = \left(\frac{1 - \cos^2 t}{\cos t}\right)^2\!.$$

By (3–14) we have
$$\left(\frac{1 - \cos^2 t}{\cos t}\right)^2 = \left(\frac{\sin^2 t}{\cos t}\right)^2 = \frac{\sin^4 t}{\cos^2 t}.$$

The right side of the given identity may be reduced as follows. By definition,
$$\tan t = \frac{\sin t}{\cos t}.$$

Hence, we have
$$\tan^2 t(1 - \cos^2 t) = \frac{\sin^2 t}{\cos^2 t} \cdot \sin^2 t = \frac{\sin^4 t}{\cos^2 t}.$$

Since both sides reduce to the same expression, $\dfrac{\sin^4 t}{\cos^2 t}$, the identity is established.

EXERCISE 3–2

Prove each of the following identities:

1. $\sin t - \cos t \tan t = 0.$

2. $\dfrac{\sin t}{\csc t} + \dfrac{\cos t}{\sec t} = 1.$

3. $\dfrac{\sin t}{\cos t} - \dfrac{\sec t}{\csc t} = 0.$

4. $\tan t \csc t = \sec t.$

5. $\sin t(\cot t + \csc t) = 1 + \cos t.$

6. $(\sin t - \cos t)^2 = 1 - 2 \sin t \cos t.$

7. $\sec^2 t + 2 \tan t = (1 + \tan t)^2.$

8. $\sin t(\csc t - \sin t) = \cos^2 t.$

9. $\tan^2 t(\cot^2 t - \cos^2 t) = \cos^2 t.$

10. $\cos^2 t + \cos^2 t \tan^2 t = 1.$

11. $(1 - \sin t)(1 + \sin t) = \cos^2 t.$

12. $(\sec t - 1)(\sec t + 1) = \tan^2 t.$

13. $\dfrac{\sin t}{1 - \cos^2 t} = \csc t.$

14. $\sin t \tan t + \cos^2 t \sec t = \sec t.$

15. $\sin^2 t \csc^2 t = 2 - \cos^2 t \sec^2 t.$

16. $\sin^4 t \sec^2 t \csc^2 t = \tan^2 t.$

17. $\tan t + \cot t = \sec t \csc t.$

18. $\csc^2 t + \sec^2 t = \sec^2 t \csc^2 t.$

19. $\sec^4 t - \sec^2 t = \tan^4 t + \tan^2 t.$

20. $\sin^4 t - \cos^4 t = \sin^2 t - \cos^2 t.$

21. $\dfrac{\sin t \tan t}{\cos^2 t - 1} = -\sec t.$

22. $\tan^2 t + \cot^2 t = \sec^2 t \csc^2 t - 2.$

23. $\dfrac{1 - \sin t}{\cos t} + \dfrac{1 - \cos t}{\sin t} = \dfrac{\sin t + \cos t - 1}{\sin t \cos t}.$

24. $\dfrac{\sec^2 t - \tan^2 t + \tan t}{\sec t} = \sin t + \cos t.$

25. $\sec^2 t = \csc^2 t(\sec^2 t - 1).$

26. $(\tan t + \cot t)^2 = \sec^2 t \csc^2 t.$

27. $1 + \tan^2 t = (\sec^2 t - 1)\csc^2 t.$

28. $\sin t(1 + \tan^2 t) - \sin t = \tan^3 t \cos t.$

29. $\dfrac{\sin t - \cos t}{\sin t + \cos t} = \dfrac{\tan t - 1}{\tan t + 1}.$

30. $\dfrac{\sin^3 t + \cos^3 t}{\sin t + \cos t} = 1 - \sin t \cos t.$

31. $\cot^2 t - \cos^2 t = \cos^2 t \cot^2 t.$

32. $\dfrac{1 - \cos t}{1 + \cos t} = (\csc t - \cot t)^2.$

3–4. TABLES OF TRIGONOMETRIC FUNCTIONS

Exact numerical values of trigonometric functions in general cannot be found. However, by use of methods beyond the scope of this book, the values can be computed to as many decimal places as desired. The results of such computations have been tabulated and are included in this text in the form of tables of trigonometric functions. Table I at the end of the text contains the values of the

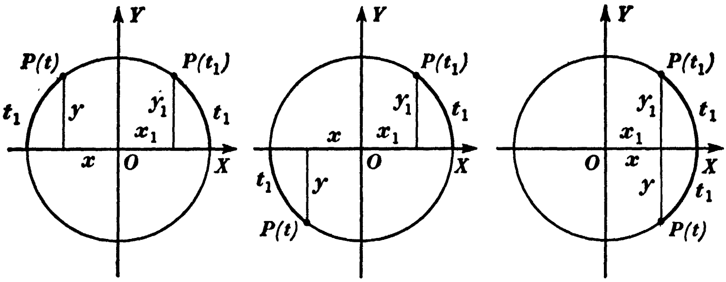

FIG. 3–6.

six functions corresponding to numbers t such that $0 \leqq t \leqq \pi/2$. Actually, since $\pi/2 = 1.5708$ approximately, the table contains values of t between 0 and 1.60.

Let $P(t) = (x, y)$ be the point corresponding to a given value of t, Fig. 3–6, and let t_1 denote the length of the shorter arc which joins $P(t)$ to the x-axis. In each case in Fig. 3–6, the point $P(t_1)$ is located by measuring the arc t_1 counterclockwise from the positive half of the x-axis. We shall call t_1 the *reference number*, or *related number*, for t. Note that $0 \leqq t_1 \leqq \pi/2$. Since t_1 is a real number, there is associated with it a point $P(t_1) = (x_1, y_1)$. Also, since t_1 lies between 0 and $\pi/2$, $P(t_1)$ must lie in the first quadrant.

In each case in Fig. 3–6 the coordinates of $P(t_1)$ must be numerically equal to those of $P(t)$; that is, $|x| = x_1$ and $|y| = y_1$. Since all the trigonometric functions are defined in terms of x and y, we can say that

(3–17) $$|\text{ any function of } t\ | = \text{same function of } t_1.$$

These functional values may not have the same algebraic sign, since $P(t_1)$ lies in the first quadrant and all functions of t_1 have positive values, whereas $P(t)$ may lie in any quadrant and the functions of t do not necessarily have positive values. It is important to see that the proper sign is chosen to make the equation a true one. The algebraic sign in each case depends on the quadrant in which $P(t)$ lies.

The following examples and Fig. 3–7 will illustrate the method of reducing a function of any positive or negative t to the same function of the reference number t_1.

We shall limit our discussion for the present to direct use of Table I and consider only values of t_1 which are shown there. The process of using the table for values of t_1 which are not shown will be treated in Section 3–10 when we discuss *interpolation*. For simplicity at this time, we shall use the approximate value $\pi = 3.14$.

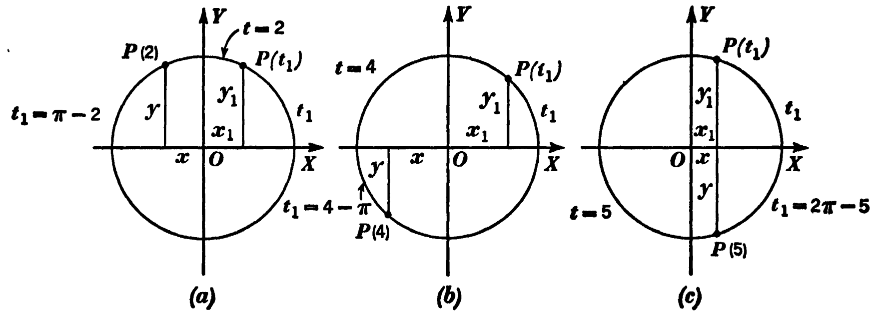

FIG. 3–7.

Example 3–5. Reduce the functions of $t = 2$ to functions of its reference number.

Solution: As shown in Fig. 3–7(*a*), $P(2)$ is in the second quadrant, and the reference number t_1 is $\pi - 2$, or 1.14. The numerical values of the functions of 1.14 may be found from Table I. The signs of the functions of 2 are determined by noting that only the sine and cosecant are positive in the second quadrant. Thus, we have:

$$\begin{aligned}
\cos 2 &= -\cos(\pi - 2) = -\cos 1.14 = -0.4176,\\
\sin 2 &= \sin(\pi - 2) = \sin 1.14 = 0.9086,\\
\tan 2 &= -\tan(\pi - 2) = -\tan 1.14 = -2.176,\\
\cot 2 &= -\cot(\pi - 2) = -\cot 1.14 = -0.4596,\\
\sec 2 &= -\sec(\pi - 2) = -\sec 1.14 = -2.395,\\
\csc 2 &= \csc(\pi - 2) = \csc 1.14 = 1.101.
\end{aligned}$$

Example 3–6. Find tan 4.

Solution: As shown in Fig. 3–7(*b*), the reference number t_1 is $4 - \pi = .86$. Since the tangent is positive in the third quadrant, $\tan 4 = \tan .86 = 1.162$.

Example 3–7. Find cos 5.

Solution: Here, as shown in Fig. 3–7(*c*), $t_1 = 2\pi - 5 = 1.28$; and $\cos 5 = \cos 1.28 = 0.2867$.

Example 3–8. Find cot 20.

Solution: To locate the point $P(20)$, we must proceed 20 units around the unit circle in a positive direction from (1, 0). The number of units in one complete revolution is $2\pi = 6.28$, and we find that

$$20 = 3(6.28) + 1.16.$$

Therefore, to locate $P(20)$, we must proceed three times around the unit circle and then continue for 1.16 additional units in a counterclockwise direction. This means that t may be taken as 1.16. Since $P(20)$ or $P(1.16)$ lies in the first quadrant, $t_1 = t = 1.16$. From the table, we have $\cot 20 = \cot 1.16 = 0.4356$.

Example 3–9. Find sin (-2).

Solution: For $t = -2$, it is shown in Fig. 3–8 that $t_1 = |-\pi - (-2)| = |-\pi + 2| = 1.14$. Hence, t_1 is the same as t_1 in Example 3–5. Since sin t is negative in the third quadrant, we have

$$\sin(-2) = -\sin 2 = -\sin 1.14 = -0.9086.$$

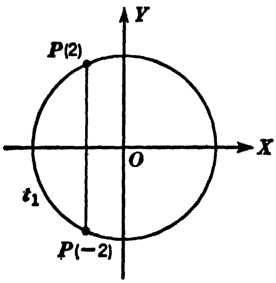

FIG. 3–8.

It should be noted that in Fig. 3–8 the points $P(-t)$ and $P(t)$ are located on opposite sides of the x-axis, and are joined by a line segment which is bisected perpendicularly by the axis. Hence the coordinates of the two points are numerically equal, but the y-coordinates have opposite signs. Therefore, by the definitions of the functions given in Section 3–2, it follows that

$$\sin(-t) = -\sin t, \qquad \csc(-t) = -\csc t,$$
$$\cos(-t) = \cos t, \qquad \sec(-t) = \sec t,$$
$$\tan(-t) = -\tan t, \qquad \cot(-t) = -\cot t.$$

Hence, if $t > 0$,

(3–18) $\qquad |\text{any function of } (-t)| = |\text{same function of } t|.$

However, the algebraic sign of the function is changed for all functions except the cosine and the secant.

Because cos t and sec t remain unchanged when t is replaced by its negative, these functions are called *even functions.* The remaining functions are called *odd functions,* since their values change sign when t is replaced by its negative.

EXERCISE 3–3

1. Construct a figure, locating each of the following points. Show the point $P(t)$ and its related number t_1. (Use $\pi = 3.14$).

a. $P(1)$. **b.** $P(3)$. **c.** $P(10)$. **d.** $P(-5)$. **e.** $P(-4)$. **f.** $P(3/2)$.

2. With the aid of Table I find each of the following values, using $\pi = 3.14$.

a. sin 1.45. **b.** cos 3.5. **c.** sec 4.75.
d. tan 5. **e.** csc (-2.41). **f.** cot (-4.50).
g. sin 28. **h.** cos 60. **i.** tan (-30).
j. $\sec \frac{3\pi}{4}$. **k.** $\cot \frac{13\pi}{4}$. **l.** $\sin \frac{22\pi}{3}$.

3–5. POSITIVE AND NEGATIVE ANGLES AND STANDARD POSITION

We have defined each of the six trigonometric functions as a relation between two sets of numbers, employing as the independent

variable a real number t whose absolute value represents the length of an arc of a unit circle. Now we shall return to the traditional viewpoint and consider trigonometric functions of angles.

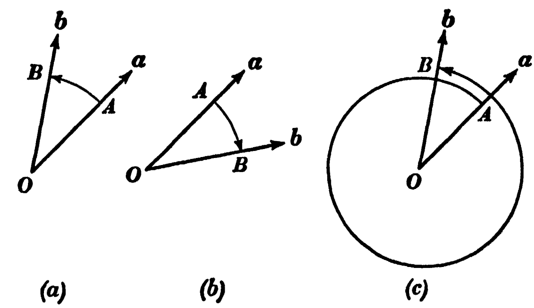

FIG. 3–9.

Although the student is probably familiar with the idea of angle from the study of geometry, we shall try to make the definition more precise. Let us select a point O in a plane and draw the half-line or ray a emanating from O, as shown in Fig. 3–9. We shall call O the *vertex* of the ray. Finally, we let A be a point on the ray in its initial position.

Now rotate the ray a about O to some terminal position b, so that the point A moves along the arc indicated by the curved arrow AB. The ray may be rotated in the counterclockwise sense, as in Fig. 3–9(a), or in the clockwise sense, as in view (b). Moreover, it may be turned through one or more complete revolutions, as in view (c). We shall speak of the position b as the terminal ray b. We have then an ordered pair of half-lines consisting of the initial ray a and the terminal ray b. We can now define an angle as follows:

An *angle* is a geometric figure consisting of two ordered rays emanating from a common vertex.

With each angle is associated a number, called the *measure* of the angle, which indicates the sense and amount of rotation required to turn from the initial ray of the angle to the terminal ray. This rotation is usually represented graphically by a curved arrow. Its evaluation will be considered in Section 3–6.

We may designate the angle in Fig. 3–9 as angle AOB; or we may use a Greek letter, such as θ, ϕ, α, β, or γ, as the designation. The line OA is called the *initial side* of angle AOB, and OB is the *terminal side*. Counterclockwise rotation, as in Fig. 3–9(a) or 3–9(c), gives rise to a *positive* angle, while clockwise rotation, such as the one in Fig. 3–9(b), gives rise to a *negative* angle.

Finally, we shall say that an angle is in *standard position* with respect to a rectangular coordinate system when its vertex is at the origin and its initial side coincides with the positive x-axis. See

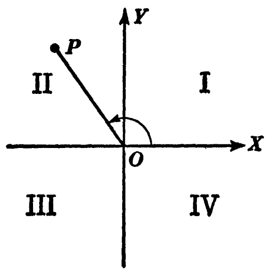

FIG. 3–10.

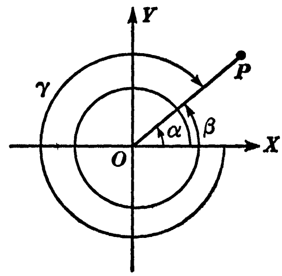

FIG. 3–11.

Fig. 3–10. When an angle is placed in standard position, the terminal side determines the quadrant to which an angle is said to belong. Thus, angle XOP in Fig. 3–10 is positive because it is generated in a counterclockwise direction, and is a second-quadrant angle because the terminal side OP lies in the second quadrant.

We note that the definition of angle does not specify that the rotation should stop at the first arrival at the terminal side OP, Fig. 3–10. In fact, angles of any size may be generated, since any number of angles which end at the terminal side OP of a given angle may be obtained simply by adding a number of complete rotations, positive or negative, to the given angle. For example, the same terminal side may also be reached by rotation in the opposite direction. All angles which are in standard position and have the same terminal sides are called *coterminal angles.*

In Fig. 3–11 the angle α is generated by rotation of OX counterclockwise to the position OP. The angle β, which is coterminal with α, is generated by adding to α one complete rotation of OX. The angle γ is a negative angle, which is coterminal with α and is generated by rotating OX in the clockwise direction to the position OP.

3–6. MEASUREMENT OF ANGLES

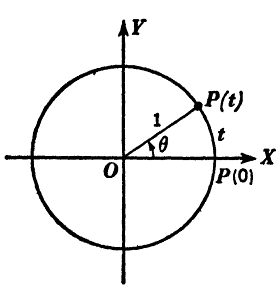

FIG. 3–12.

The problem of measuring an angle is equivalent to that of finding the measure of the associated arc. One should, therefore, apply the discussion of Section 3–1 and construct a unit circle as shown in Fig. 3–12.

Let θ be an angle in standard position. Since the initial and terminal sides of the angle intersect the circle in the points $P(0)$ and $P(t)$, respectively, the problem of measuring the angle θ reduces to that of measuring the appropriate

arc length t. Thus, the measure of the angle can be found in terms of a real number in any one of several ways, depending on the unit of measure chosen.

We shall consider first the *circular system,* or *natural system,* of measuring angles, which is used almost exclusively in the calculus and its applications. Its fundamental unit is the radian. This unit may be defined as follows:

A *radian* is the measure of an angle which, if placed at the center of a circle, intercepts an arc on the circumference equal in length to the radius of the circle.

In Fig. 3–13 the angle AOB is 1 radian, and the length of the subtended arc AB is equal to the radius r.

If the circle selected for measuring a radian is a unit circle, we have an alternate definition of a radian. That is, a radian is an angle which intercepts a unit arc on a unit circle.

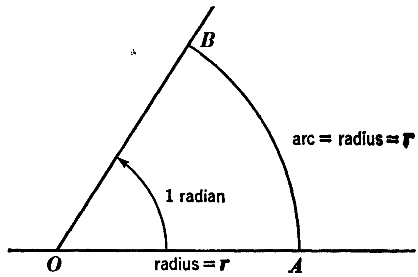

FIG. 3–13.

Another system of measuring angles is the *sexagesimal system,* or *degree system,* which is commonly used in ordinary calculations involving angles. The fundamental unit of this system is the *degree.*

In Section 3–7 we shall study various relations between radians and degrees, and shall develop rules which allow us to convert from one system to the other.

In discussing angles, we frequently use the term *angle,* in place of *measure of an angle,* and we rely on the context to make the meaning clear. Thus, when we say "$\theta = 2$," we mean, "θ is an angle whose measure is 2 radians." The word *radian* is usually omitted when an angle is expressed in terms of radians.

3–7. THE RELATION BETWEEN RADIANS AND DEGREES

Since an arc that is equal in length to the radius of a circle subtends an angle of one radian at the center, it follows that the whole circumference, which is 2π times the radius, subtends an angle of 2π radians. Furthermore, the whole circumference subtends a central angle of 360°. Therefore,

$$2\pi \text{ radians} = 360°,$$

and

$$\pi \text{ radians} = 180°.$$

If the approximate value 3.1416 is used for π,

$$1 \text{ radian} = \frac{180°}{\pi} = \frac{180°}{3.1416} \text{ (approximately)},$$

or

$$1 \text{ radian} = 57.29578° \text{ (approximately)},$$

or

$$1 \text{ radian} = 57°17'45'' \text{ (approximately)}.$$

Also,

$$1° = \frac{\pi}{180} \text{ radians} = 0.01745329 \text{ radians (approximately)}.$$

In order to make the conversion to radians easier when the angle is expressed in degrees, minutes, and seconds, we give the following values:

$$1' = 0.00029089 \text{ radians},$$

and

$$1'' = 0.00000485 \text{ radians}.$$

Therefore, one of the following rules can be used to convert from degrees to radians or from radians to degrees:

To convert from degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$, or 0.0174533.

To convert from radians to degrees, multiply the number of radians by $\frac{180}{\pi}$, or 57.29578.

Note, however, that certain angles are commonly expressed in terms of π radians, in order to avoid approximate values. For example,

$$180° = \pi \text{ radians}, \qquad 45° = \pi/4 \text{ radians},$$
$$90° = \pi/2 \text{ radians}, \qquad 30° = \pi/6 \text{ radians}.$$

3–8. ARC LENGTH AND AREA OF A SECTOR

In Fig. 3–14 is shown a circle of radius r. In such a circle an angle at the center equal to one radian subtends an arc on the circumference equal to r. Similarly, by the definition of a radian, the number of units in the arc s intercepted by a central angle equal to θ radians is given by the relationship

$$\frac{s}{\theta} = \frac{r}{1}.$$

Thus,

(3–19) $$s = r\theta,$$

or

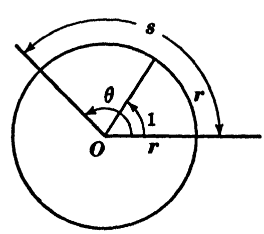

FIG. 3–14.

$$\text{arc} = (\text{radius}) \cdot (\text{central angle expressed in radians}).$$

Now let A denote the area of the sector bounded by two radii and an arc of length s. If θ is the number of radians in the central angle of the sector, then the ratio of the area A of the sector to the area of the whole circle, or πr^2, equals the ratio of the angle θ to the angle in the whole circle, or 2π. That is,

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi},$$

or

$$A = \frac{1}{2} r^2 \theta. \tag{3–20}$$

If the central angle of an arc or a sector is expressed in degrees, it must be re-expressed in radians before (3–19) or (3–20) can be applied.

Example 3–10. Express 210° in terms of π radians.

Solution: Since $1° = \frac{\pi}{180}$ radians, $210° = 210 \cdot \frac{\pi}{180} = \frac{7\pi}{6}$ radians. Thus, $210° = \frac{7\pi}{6}$ radians.

Example 3–11. Express 12°15′20″ in radians.

Solution: Multiply the decimal parts of a radian given in Section 3–7 for 1°, 1′, and 1″ by 12, 15, and 20, respectively. The results are as follows:

$$\begin{aligned} 12°\ 0'\ 0'' &= .20943948 \text{ radians} \\ 15'\ 0'' &= .00436335 \text{ radians} \\ 20'' &= .00009700 \text{ radians} \\ \hline 12°\ 15'\ 20'' &= .21389983 \text{ radians.} \end{aligned}$$

Example 3–12. Express $\frac{5\pi}{6}$ radians in degrees.

Solution: Since π radians $= 180°$, $\frac{5\pi}{6}$ radians $= \frac{5}{6}(180°) = 150°$.

Example 3–13. Express 3.5 radians in degrees, minutes, and seconds.

Solution: First, convert the radians to degrees, as follows:

$$3.5 \text{ radians} = (3.5)\,(57.29578°) = 200.5352°.$$

To find the number of minutes, multiply the decimal part of a degree, or 0.5352, by 60. Thus, $0.5352° = (60)\,(0.5352)$ minutes $= 32.112'$.

To find the number of seconds, multiply the decimal part of a minute, or 0.112, by 60. The result is $0.112' = (60)\,(0.112)$ seconds $= 6.72''$.

Hence, 3.5 radians $= 200°32'6.72''$.

Example 3–14. The radius of a circle is 5 inches. Find the length of the arc of the circle subtended by a central angle of 30°.

Solution: Since $30° = \frac{\pi}{6}$, the central angle θ is $\frac{\pi}{6}$. Also, $r = 5$. Therefore, by (3–19),

$$s = r \cdot \theta = 5 \cdot \frac{\pi}{6} = \frac{5}{6}(3.1416) = 2.618 \text{ inches.}$$

Example 3–15. In a circle of radius 6 inches, what is the area of a sector whose central angle is 60°?

Solution: By (3–20), the area of the sector is $\frac{1}{2}r^2\theta$. Since $\theta = 60° = \frac{\pi}{3}$,

$$A = \frac{1}{2}(36)\frac{\pi}{3} = 6\pi \text{ square inches.}$$

EXERCISE 3–4

In each problem from 1 to 25, express the given angle in radians.

1. 60°.	**2.** 45°.	**3.** 30°.	**4.** 10°.	**5.** 120°.
6. 150°.	**7.** 12°.	**8.** 90°.	**9.** 240°.	**10.** 330°.
11. 72°.	**12.** 20°.	**13.** 215°.	**14.** 196°.	**15.** 321°.
16. 283°.	**17.** 63°.	**18.** 30°10′.	**19.** 46°21′.	**20.** 236°37′.
21. 82°16′.	**22.** 63°21′17″.	**23.** 183°57′43″.	**24.** 392°44′27″.	**25.** 93°31′38″.

In each problem from 26 to 40, express the given angle in degrees.

26. $\pi/6$.	**27.** $\pi/4$.	**28.** $\pi/8$.	**29.** $3\pi/2$.	**30.** $4\pi/5$.
31. $\pi/12$.	**32.** $5\pi/18$.	**33.** $7\pi/2$.	**34.** $5\pi/3$.	**35.** $3\pi/20$.
36. 3.7 rad.	**37.** 8.21 rad.	**38.** 0.34 rad.	**39.** 0.763 rad.	**40.** 0.8136 rad.

In each problem from 41 to 56, draw the given angle in standard position and indicate its terminal side.

41. 30°.	**42.** $\pi/4$.	**43.** $\pi/3$.	**44.** 90°.
45. $22\frac{1}{2}°$.	**46.** $2\pi/3$.	**47.** 170°.	**48.** $17\pi/18$.
49. $-\pi/2$.	**50.** 630°.	**51.** 360°.	**52.** -4π.
53. $7\pi/3$.	**54.** 1000°.	**55.** $9\pi/4$.	**56.** $-11\pi/6$.

57. In a circle of radius 4 feet, find the length of the arc intercepted by an angle of $7\pi/6$ radians. Find the angle in radians that intercepts a 5-foot arc.

58. A central angle in a circle of radius 15 inches intercepts an arc of 5 inches. Find the number of radians in the central angle. Express this angle in degrees and minutes, rounding off the result to the nearest minute.

59. A central angle of 62°14′ intercepts an arc of 16 inches on the circumference of a circle. Find the radius of the circle.

60. Find the area of a circular sector whose radius is 7 inches and whose central angle is *a*) 4 radians; *b*) 75°; *c*) 3 radians.

61. The area of a circular sector is 72 square inches. Find the angle if the radius is *a*) 6 inches; *b*) 9 inches; *c*) 5 feet.

62. The area of a circular sector is 126 square inches. Find the radius if the angle is *a*) 128°; *b*) 1.6 radians; *c*) 30°.

3–9. TRIGONOMETRIC FUNCTIONS OF ANGLES

Let θ be an angle in standard position, as shown in Fig. 3–15. With θ we can associate a real number t, which is the measure of the angle in radians. This concept is equivalent to our previous concept of t, when t was interpreted as the length of an arc laid off on the unit circle by starting at the point $(1, 0)$ and terminating at

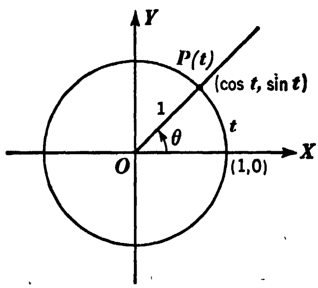

FIG. 3–15.

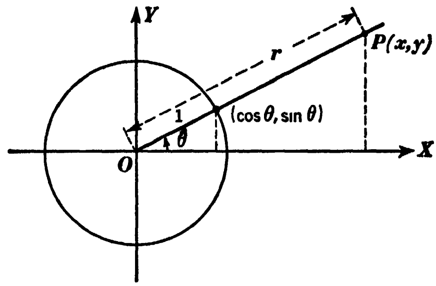

FIG. 3–16.

the point $P(t)$. Such an association of the angle θ with the directed length t of an arc of a unit circle allows us to define the cosine and sine of θ as $\cos\theta = \cos t$ and $\sin\theta = \sin t$. A similar procedure may be followed for the other functions of θ.

Now consider Fig. 3–16, where we show an angle θ in standard position and a unit circle. By the definition of θ, the terminal side of θ intersects the unit circle at the point $(\cos\theta, \sin\theta)$. This is, of course, the point designated previously as $P(t)$. We now extend the terminal side of θ to an arbitrary point P with coordinates (x, y). The length of the radius vector OP is $r = \sqrt{x^2 + y^2}$.

If we drop perpendiculars from the points $(\cos\theta, \sin\theta)$ and (x, y) to the x-axis, the right triangles thus constructed are similar. Therefore,

$$\frac{x}{r} = \frac{\cos\theta}{1} \quad \text{and} \quad \frac{y}{r} = \frac{\sin\theta}{1}.$$

Hence, the coordinates of the point $P(x, y)$ on the terminal side are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Using these results with the definitions of the functions from Section 3–2, we can express the values of the six functions in terms of x, y, and r. Thus,

$$\begin{aligned} \sin\theta &= y/r, & \csc\theta &= r/y,\\ \cos\theta &= x/r, & \sec\theta &= r/x,\\ \tan\theta &= y/x, & \cot\theta &= x/y. \end{aligned} \tag{3–21}$$

3–10. TABLES OF NATURAL TRIGONOMETRIC FUNCTIONS OF ANGLES

Tables of natural trigonometric functions are so labeled to distinguish them from tables of the logarithms of these functions.

Angles in Radians. In Section 3–4 Table I was used to find values of trigonometric functions of the type cos 2 or sin $2\pi/3$. On the basis of the definitions of the functions of an angle given in Section 3–9, Table I may also be used to find the functions of angles measured in radians.

Example 3–16. Find the cosine of an angle of 1.43 radians.

Solution: From Table I, $\cos 1.43 = 0.1403$.

Angles in Degrees. Table II at the end of this text contains the approximate values of the six functions of acute angles expressed in degrees and minutes. It is a four-place table of the functions of angles at intervals of 10 minutes.

To find the value of a function of an angle between 0° and 45°, first locate the angle in one of the columns *at the left,* and then look for the value on the same line in the column *headed* by the name of the desired function. For an angle between 45° and 90°, locate the angle in a column *at the right,* and then look for the value on the same line in the column with the name of the desired function *at its foot.*

Table II should be referred to in working through these illustrative examples.

Example 3–17. Find sin 32°40′.

Solution: This angle is between 0° and 45°. Look in the left-hand column to find 32°40′, and then go to the right to the column headed *sin*. There find 0.5398. Hence,

$$\sin 32°40' = 0.5398.$$

Example 3–18. Find cos 56°20′.

Solution: This angle is between 45° and 90°. So look in the right-hand column to find 56°20′, noting that 56°20′ is above 56°00′, and then go to the left to the column with *cos* at its foot. Thus, $\cos 56°20' = 0.5544$.

The following examples illustrate the procedure for finding an angle corresponding to a given value of a function.

Example 3–19. Given $\tan\theta = 4.511$, find θ, assuming that $0° \leqq \theta \leqq 90°$.

Solution: Since $\tan\theta$ is greater than 1, θ is greater than 45°. Therefore, search through the columns marked *tan* at the foot for the given number 4.511. The corresponding angle in the right-hand column is 77°30′. So $4.511 = \tan 77°30'$, or $\theta = 77°30'$.

Example 3–20. Given $\cos\theta = 0.8660$, find θ, assuming that $0° \leqq \theta \leqq 90°$.

Solution: By looking through the columns with *cos* at either the head or the foot, find 0.8660. Since this value is in a column headed *cos*, use the left-hand column for the corresponding angle, which is 30°. Hence, $\theta = 30°$.

Interpolation. When either the given angle or the given value of a function is not printed in the table, we can find the desired value or angle by using a method of approximation known as *interpolation.* We assume that the change in the value of the function is directly proportional to the change in the angle. Although this assumption is not strictly valid, it gives values that are accurate enough for many practical purposes if we limit its use to small changes in the angle.

The process of *direct interpolation* is used if the angle is given and we need to find the value of either an increasing function of the angle, such as the sine, or a decreasing function, such as the cosine. *Inverse interpolation* is used when the value of a trigonometric function is known and the angle is to be found.

Example 3–21. Find $\sin 18°12'$.

Solution: This angle is not listed in the table, but it lies between 18°10′ and 18°20′. From the table we find that

$$\sin 18°10' = 0.3118,$$

and

$$\sin 18°20' = 0.3145.$$

The desired value of $\sin 18°12'$ will then lie between 0.3118 and 0.3145.

The tabular difference, that is, the difference between the two values listed in the table, is 0.0027. Also, the difference between the angles 18°10′ and 18°20′ is 10′, while the angle 18°12′ differs from 18°10′ by 2′. Since the change in the angle from 18°10′ to 18°12′ is 2/10 of the change from 18°10′ to 18°20′, we assume that the corresponding change in the value of the sine will be $(0.2)\,(0.0027) = 0.0005$, and the amount to be added to 0.3118 is 0.0005. Hence, $\sin 18°12' = 0.3123$.

The accompanying diagrammatic arrangement presents this same operation in tabular form:

$$10\left\{\begin{array}{l} 2\left\{\begin{array}{l}\sin 18°10' = 0.3118\\ \sin 18°12' = 0.3118 + x\end{array}\right\} x \\ \sin 18°20' = 0.3145\end{array}\right\} 0.0027$$

Since the angle 18°12′ is 2/10 of the way from 18°10′ to 18°20′, the corresponding functional value will be 2/10 of the way from 0.3118 to 0.3145. Therefore,

$$\frac{x}{0.0027} = \frac{2}{10},$$

or

$$x = (0.2)\,(0.0027) = 0.0005.$$

This amount is to be added to 0.3118. Hence, the value of the function is 0.3123, and sin 18°12′ = 0.3123.

Example 3–22. Find cos 73°48′.

Solution: The process is similar to that in Example 3–21. However, since the cosine decreases as the angle increases, we subtract 8/10 of the tabular difference from cos 73°40′. We find the values of cos 73°40′ and cos 73°50′ in a column of the table labeled *cos* at the bottom. The work may be indicated as follows:

$$10\left\{\begin{array}{l} 8\left\{\begin{array}{l}\cos 73°40' = 0.2812 \\ \cos 73°48' = 0.2812 - x\end{array}\right\}x \\ \cos 73°50' = 0.2784\end{array}\right\}0.0028$$

$$\frac{x}{0.0028} = \frac{8}{10},\ \text{or}\ x = (0.8)\,(0.0028) = 0.0022.$$

Hence, the amount to be subtracted from 0.2812 is 0.0022, and cos 73°48′ = 0.2790.

The inverse process of finding the angle when the given value of a function is not printed in the table is performed in a similar fashion. Here, since we know the value of the function, we find the two values in the table nearest the given value, one less than it and one greater. Again making the assumption that small changes in the value of the function are proportional to small changes in the angle, we proceed as indicated in the following example.

Example 3–23. Find θ if cot θ = 0.8780.

Solution: This value of the cotangent is not in the table but lies between the entries 0.8796 and 0.8744. To these correspond, respectively, the angles 48°40′ and 48°50′. We have, therefore, the following tabulation:

$$10\left\{\begin{array}{l} x\left\{\begin{array}{ll}\cot 48°40' & = 0.8796 \\ \cot 48°\,(40 + x)' & = 0.8780\end{array}\right\}0.0016 \\ \cot 48°50' \qquad\qquad = 0.8744\end{array}\right\}0.0052$$

$$\frac{x}{10} = \frac{16}{52},\ \text{and}\ x = 3.$$

Hence, θ = 48°43′.

EXERCISE 3–5

In each of the problems from 1 to 30, use Table II to find the value of the given function. Interpolate whenever necessary.

1. sin 36°20′.
2. cot 128°40′.
3. sec 23°40′.
4. cos 96°50′.
5. sin 132°10′.
6. tan (– 28°10′).
7. csc 223°30′.
8. sec 39°30′.
9. cot 283°50′.
10. sin 98°40′.
11. cos 75°30′.
12. cot (– 133°30′).
13. sec (– 392°10′).
14. csc (– 416°20′).
15. tan 623°40′.
16. tan 298°52′.
17. cot 55°43′.
18. csc (– 44°51′).
19. sin 57°32′.
20. cot 3°16′.
21. sin (– 280°33′).
22. cot 28°01′.
23. tan 27°16′.
24. csc (– 245°29′).
25. cos (– 72°58′).
26. sin 312°37′.
27. tan 636°02′.
28. sin (– 16°47′).
29. csc 289°06′.
30. cos 126°19′.

In each of the problems from 31 to 60, use Table II to find the values of θ between 0° and 360° which satisfy the given equation. Express the results to the nearest minute, interpolating whenever necessary.

31. $\tan\theta = -0.1198$.
32. $\sin\theta = 0.3062$.
33. $\cot\theta = 1.091$.
34. $\cos\theta = 0.7951$.
35. $\tan\theta = 0.0553$.
36. $\sin\theta = 0.2419$.
37. $\sin\theta = 0.5783$.
38. $\cot\theta = -0.6494$.
39. $\tan\theta = 1.511$.
40. $\cos\theta = -0.4147$.
41. $\sin\theta = -0.9959$.
42. $\cot\theta = 0.0437$.
43. $\tan\theta = 8.345$.
44. $\cot\theta = -0.3121$.
45. $\tan\theta = 1.446$.
46. $\sin\theta = 0.6702$
47. $\tan\theta = 0.9043$.
48. $\cot\theta = 2.398$.
49. $\cos\theta = 0.9503$.
50. $\cos\theta = -0.5090$.
51. $\cos\theta = 0.8519$.
52. $\cot\theta = -1.381$.
53. $\cot\theta = 0.4230$.
54. $\sin\theta = 0.2491$.
55. $\cot\theta = 7.000$.
56. $\tan\theta = -0.1191$.
57. $\cos\theta = -0.1323$.
58. $\cot\theta = 0.1340$.
59. $\tan\theta = -18.00$.
60. $\tan\theta = 3.235$.

In each of the problems from 61 to 72, find the value of the given function. Interpolate whenever necessary. Take π as 3.14.

61. sin 0.93.
62. cot 2.46.
63. sec (– 1.24).
64. tan 8.71.
65. csc 9.43.
66. cot 0.678.
67. tan 0.333.
68. $\cot\left(-\frac{\pi}{4}\right)$
69. $\cos\frac{3\pi}{4}$.
70. $\sin\left(-\frac{7\pi}{2}\right)$.
71. csc 0.968.
72. cot (– 0.643).

In each of the problems from 73 to 84, find the values of θ, in radians, between 0 and 2π which satisfy the given equation. Use Table I and express the results to three decimal places, interpolating whenever necessary.

73. $\cos\theta = 0.9759$.
74. $\sin\theta = 0.9967$.
75. $\tan\theta = 2.066$.
76. $\sec\theta = 2.563$.
77. $\tan\theta = 0.9413$.
78. $\sin\theta = -0.7360$.
79. $\cos\theta = 0.4010$.
80. $\tan\theta = 1.6$.
81. $\sin\theta = 0.91$.
82. $\cot\theta = 0.39$.
83. $\cos\theta = 0.84$.
84. $\cot\theta = 1.031$.

4 The Laws of Exponents

4–1. POSITIVE INTEGRAL EXPONENTS

When studying the progress of algebra up to the sixteenth century, one cannot help but be perplexed by either the total absence of symbolism or, when present, the lack of uniformity in its use. At first, unknown quantities were often represented by words. Later, symbols made from abbreviations and initial letters of these words were used to indicate mathematical concepts, such as number, power, and square.

Descartes (1637) is generally credited with our present system of exponents. He introduced the Hindu-Arabic numerals as exponents, using the notations a, aa [sic], a^3, a^4, etc. The writing of a repeated letter for the second power of the unknown continued for many years.

Laws for positive integral exponents were introduced in Section 1–11, without proofs. We shall now establish these laws and extend them to apply also to zero, negative, and fractional exponents.

We recall that if n is any positive integer, a^n means the product of n factors each equal to a. In this notation, a is the *base* and n is the *exponent* or *power*. We shall proceed to establish the following laws for positive integral exponents.

Law of Multiplication. If a is a real number, and if m and n are positive integers,

$$a^m \cdot a^n = a^{m+n}. \tag{4–1}$$

Proof. Proof of this relationship follows from the definition of a^n and the associative law for multiplication. Thus,

$$a^m = a \cdot a \cdot \cdots \cdot a \qquad \text{(to } m \text{ factors)},$$

and

$$a^n = a \cdot a \cdot \cdots \cdot a \qquad \text{(to } n \text{ factors)}.$$

Hence,

$$\begin{aligned} a^m \cdot a^n &= [a \cdot a \cdot \cdots \cdot a(\text{to } m \text{ factors})]\,[a \cdot a \cdot \cdots \cdot a(\text{to } n \text{ factors})] \\ &= a \cdot a \cdot \cdots \cdot a(\text{to } m + n \text{ factors}) \\ &= a^{m+n}. \end{aligned}$$

For example, $x^3 \cdot x^5 = x^8$, and $y^k \cdot y^{k+3} = y^{2k+3}$.

Law of Division. If a is a non-zero real number, and if m and n are positive integers such that $m > n$, then

$$\frac{a^m}{a^n} = a^{m-n}. \tag{4-2}$$

If $a \neq 0$, and if $n > m$, then

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}. \tag{4-3}$$

Proof. Proofs of these relationships follow:

If $m > n$, then $m - n$ is positive. By (4–1),

$$a^{m-n} \cdot a^n = a^{(m-n)+n} = a^m.$$

Hence, dividing both sides by a^n, we have

$$a^{m-n} = \frac{a^m}{a^n}.$$

For example,

$$\frac{x^8}{x^5} = x^3, \quad \text{and} \quad \frac{3^7}{3^5} = 3^2.$$

If $m < n$, then $n - m$ is positive. By (4–1),

$$a^m \cdot a^{n-m} = a^{m+(n-m)} = a^n.$$

Divide both sides by a^{n-m} to obtain

$$a^m = \frac{a^n}{a^{n-m}}.$$

Now, dividing both sides by a^n, we have

$$\frac{a^m}{a^n} = \left(\frac{a^n}{a^{n-m}}\right) \Big/ a^n = \frac{a^n}{a^n} \cdot \frac{1}{a^{n-m}} = \frac{1}{a^{n-m}}.$$

For example,

$$\frac{x^4}{x^7} = \frac{1}{x^3}, \quad \text{and} \quad \frac{3^5}{3^7} = \frac{1}{3^2}.$$

Law for a Power of a Power. If a is a real number, and if m and n are positive integers, then

$$(a^m)^n = a^{mn}. \tag{4-4}$$

Proof. This relationship can be easily proved as follows:

By the associative laws for multiplication and addition, the law of multiplication expressed by (4–1) can be extended to three or more factors. Thus,

$$\begin{aligned} a^m \cdot a^n \cdot a^p &= (a^m \cdot a^n) \cdot a^p \\ &= a^{m+n} \cdot a^p \\ &= a^{(m+n)+p} \\ &= a^{m+n+p}. \end{aligned}$$

A similar relationship can be written for any number of factors. That is,

$$(a^m)(a^p) \cdots (a^r) = a^{m+p+\cdots+r}.$$

We may now take $m = p = \cdots = r$ to get

$$(a^m)\,(a^m)\cdots(a^m)\ (\text{to } n \text{ factors}) = a^{m+m+\cdots+m} = a^{mn}.$$

Hence,

$$(a^m)^n = a^{mn}.$$

For example,

$$(x^2)^3 = x^6, \text{ and } (2^{2k+1})^5 = 2^{10k+5}.$$

Law for a Power of a Product. If a and b are real numbers, and if m and n are positive integers, then

$$(ab)^n = a^n b^n. \tag{4-5}$$

Proof. In proving this relationship, we make use of the associative and commutative laws of multiplication. Thus,

$$\begin{aligned}(ab)^n &= (ab)\cdot(ab)\cdot\cdots\cdot(ab)\ (\text{to } n \text{ factors})\\ &= [a\cdot a\cdot\cdots\cdot a(\text{to } n \text{ factors})]\,[b\cdot b\cdot\cdots\cdot b(\text{to } n \text{ factors})]\\ &= a^n b^n.\end{aligned}$$

For example,

$$(-2x^2y^3)^5 = (-2)^5\,x^{10}y^{15} = -32x^{10}y^{15}.$$

Law for a Power of a Quotient. If a and b are real numbers, if $b \neq 0$, and if n is a positive integer, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}. \tag{4-6}$$

Proof. By applying the law for multiplying fractions, we have

$$\left(\frac{a}{b}\right)^n = \frac{a}{b}\cdot\frac{a}{b}\cdot\cdots\cdot\frac{a}{b}\ (\text{to } n \text{ factors}) = \frac{a^n}{b^n}.$$

For example,

$$\left(\frac{3xy^k}{z^2}\right)^2 = \frac{3^2x^2y^{2k}}{z^4} = \frac{9x^2y^{2k}}{z^4}.$$

An exponent affects only that quantity to which it is attached. Thus, $-5x(y^3)^2 = -5xy^6$, whereas $(-5xy^3)^2 = 25x^2y^6$.

So far we have defined a^n only when n is a positive integer. We shall now introduce zero, negative integer, and rational powers in such a way that they will obey the same laws which were proved for positive integral exponents.

4–2. MEANING OF a^0

We shall define the zero exponent by the equation

$$a^0 = 1 \qquad (a \neq 0). \tag{4-7}$$

A few illustrations are:

$$x^0 = 1, \quad 7^0 = 1, \quad 5(2xy^3)^0 = 5, \quad (a^2 - bx)^0 = 1, \quad \left(\frac{x^2}{2y^3}\right)^0 = 1.$$

If a in (4–2) is not zero and $m = n$, we get

$$\frac{a^m}{a^n} = a^{m-n} = a^0.$$

In this case, the quotient on the left equals 1, while the value of the term on the right is a^0. Since $a^0 = 1$, by definition, the law of division holds for $n = m$, as well as for $m > n$ and $n > m$.

The student should note that (4–7) gives the only possible definition of a^0 if the law expressed by (4–2) and (4–3) is to hold for the zero exponent, as can be seen from the foregoing discussion.

We shall show that the definition $a^0 = 1$ is consistent with the five laws of exponents in Section 4–1; that is, we shall show that these laws also hold when any exponent is zero. In the following explanations, where a quantity occurs in a denominator, we assume that it is not zero. Also, the exponents are assumed to be non-negative integers.

Let us, for sake of discussion, suppose that $n = 0$ in (4–1), that is, in

$$a^m \cdot a^n = a^{m+n}.$$

Then we have

$$a^m \cdot a^n = a^m \cdot a^0 = a^m \cdot 1 = a^m,$$

or

$$a^{m+n} = a^{m+0} = a^m = a^m \cdot a^n.$$

Hence, the law of multiplication holds when $n = 0$. A similar procedure will verify the law if $m = 0$.

Now let us suppose that $n = 0$ in (4–2), that is, in

$$\frac{a^m}{a^n} = a^{m-n}.$$

Then

$$\frac{a^m}{a^n} = \frac{a^m}{a^0} = \frac{a^m}{1} = a^m, \quad \text{and} \quad a^{m-n} = a^{m-0} = a^m.$$

Hence, (4–2) holds when $n = 0$.

If $m = 0$ in (4–3), we have

$$\frac{a^m}{a^n} = \frac{a^0}{a^n} = \frac{1}{a^n}, \quad \text{and} \quad \frac{1}{a^{n-m}} = \frac{1}{a^{n-0}} = \frac{1}{a^n}.$$

It is clear that in (4–2) m cannot be zero, and in (4–3) n cannot be zero. Therefore, the law of division holds.

Suppose that $n = 0$ in (4–4), that is, in

$$(a^m)^n = a^{mn}.$$

Then

$$(a^m)^n = (a^m)^0 = 1, \text{ and } a^{mn} = a^{m \cdot 0} = a^0 = 1.$$

If $m = 0$ in (4–4), we have

$$(a^m)^n = (a^0)^n = 1^n = 1, \text{ and } a^{mn} = a^{0 \cdot n} = a^0 = 1.$$

Hence, the law for a power of a power holds.

Now consider (4–5), which is

$$(ab)^n = a^n b^n.$$

If $n = 0$,

$$(ab)^n = (ab)^0 = 1, \quad \text{and} \quad a^n b^n = a^0 b^0 = 1 \cdot 1 = 1.$$

Finally, we let $n = 0$ in (4–6), that is, in

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Then

$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 = 1, \quad \text{and} \quad \frac{a^n}{b^n} = \frac{a^0}{b^0} = \frac{1}{1} = 1.$$

The demonstrations just given prove that the five laws of exponents, originally stated for positive integral powers, are true for all non-negative integral powers, and that the law of division is true even when the exponents are equal.

4–3. NEGATIVE EXPONENTS

In order to extend the meaning of exponents to negative integers, we define a^{-n} by the following relationship:

$$a^{-n} = \frac{1}{a^n} \qquad (a \neq 0), \tag{4–8}$$

where n is a positive integer.

Several illustrations are:

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25}, \quad \frac{1}{10^{-3}} = 10^3 = 1000,$$

$$(a - bx)^{-2} = \frac{1}{(a - bx)^2}, \quad \text{and} \quad \left(\frac{x}{y}\right)^{-1} = \frac{1}{\frac{x}{y}} = \frac{y}{x}.$$

As in Section 4–2, we shall show that our definition is consistent with the five laws of exponents.

Let us first note that (4–8) is true even if $n = 0$ or if n is a negative integer. If $n = 0$, then

$$a^{-n} = a^0 = 1 = \frac{1}{1} = \frac{1}{a^0} = \frac{1}{a^n}.$$

If $n = -p$, where p is a positive integer, then $a^{-n} = a^p = 1 \Big/ \frac{1}{a^p}$ $= \frac{1}{a^{-p}} = \frac{1}{a^n}.$

We shall use this result in the proofs that follow.

In order to extend (4–1), let m be a non-negative integer, and let n be a negative integer, say, $n = -p$. In this case it is also assumed that $a \neq 0$. Then

$$a^m \cdot a^n = a^m \cdot a^{-p} = a^m \cdot \frac{1}{a^p} = \frac{a^m}{a^p}.$$

If $m \geqq p$, we have, by (4-2) for non-negative exponents,

$$a^m \cdot a^n = \frac{a^m}{a^p} = a^{m-p} = a^{m+(-p)} = a^{m+n}.$$

If $m < p$, we have, by (4-3) for non-negative exponents,

$$a^m \cdot a^n = \frac{a^m}{a^p} = \frac{1}{a^{p-m}} = a^{-(p-m)} = a^{m-p} = a^{m+n}.$$

A similar demonstration establishes (4-1) in case $n \geqq 0$ and $m < 0$, or in case $m < 0$, and $n < 0$.

Proof of the extension of (4-2) rests on the validity of the law of multiplication just established. If $a \neq 0$, and if m and n are integers (positive, negative, or zero), then

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n} = a^{m-n}.$$

Although (4-3) is now an immediate consequence, it is not really needed, in view of the general validity of (4-8). The demonstration just given allows the law of division to be stated as a single relationship as follows:

$$\frac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0). \tag{4-9}$$

Thus, a single law applies, regardless of whether $m > n$, $n > m$, or $m = n$, where m and n are arbitrary integers.

Now consider the law for a power of a power. In (4-4) let $n = -p$, where $p \geqq 0$, while $m \geqq 0$. Then

$$(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} = \frac{1}{a^{mp}}.$$

Also,

$$a^{mn} = a^{m(-p)} = a^{-mp} = \frac{1}{a^{mp}}.$$

Hence, (4-4) holds in this case.

If $m = -p$, where $p \geqq 0$, while $n \geqq 0$, we have

$$(a^m)^n = (a^{-p})^n = \left(\frac{1}{a^p}\right)^n = \frac{1}{a^{np}}, \quad \text{and } a^{mn} = a^{(-p)n} = a^{-pn} = \frac{1}{a^{np}}.$$

Again (4-4) holds. If both m and n are negative, a similar procedure is used, and the extension of (4-4) holds.

To extend the law for a power of a product, let $n = -p$ in (4-5), where $p \geqq 0$. Then

$$(ab)^n = (ab)^{-p} = \frac{1}{(ab)^p} = \frac{1}{a^p b^p}, \quad \text{and} \quad a^n b^n = a^{-p} b^{-p} = \frac{1}{a^p} \cdot \frac{1}{b^p} = \frac{1}{a^p b^p}.$$

So (4-5) is verified for negative integral values of n.

To verify the extended law for a power of a quotient, assume that $n = -p$ in (4-6), where $p \geqq 0$. Then

$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^{-p} = \frac{1}{\left(\frac{a}{b}\right)^p} = 1 \Big/ \frac{a^p}{b^p} = \frac{b^p}{a^p}, \quad \text{and} \quad \frac{a^n}{b^n} = \frac{a^{-p}}{b^{-p}} = \frac{b^p}{a^p}.$$

Hence, (4–6) holds for negative integral values of n.

Thus, the laws of exponents hold for positive integral exponents, zero exponents, and negative integral exponents. In Section 4–5 we shall consider the case of fractional exponents.

From the general validity of (4–8), it follows immediately that a factor of the numerator or the denominator of a fraction can be moved from the numerator to the denominator, or vice versa, provided only that we change the sign of its exponent. For example,

$$a^2b^{-3} = \frac{a^2}{b^3}, \quad \text{and} \quad \frac{x^3}{yz^{-2}} = \frac{x^3z^2}{y}.$$

4–4. SCIENTIFIC NOTATION

We are now in a position to introduce certain simplifications when operating with very large or very small numbers, as are customarily used in scientific writing. Any positive number that is greater than 10 or less than 1 may be written compactly by expressing it in *standard form,* that is, by writing it as a number that lies between 1 and 10 multiplied by a suitable positive or negative integral power of 10. Thus, 27,000 would be written $2.7 \cdot 10^4$. Similarly, 0.00031 would be $3.1 \cdot 10^{-4}$.

Example 4–1. The speed of light is 186,000 miles per second. Express this number in scientific notation.

Solution: The given number 186,000 may be written as $1.86 \cdot 10^5$.

Example 4–2. The rest mass of an electron is $9.11 \cdot 10^{-28}$ grams. How many zeros would be required between the decimal point and the first non-zero digit, 9, if the number were written in decimal notation?

Solution: The exponent -28 means that we would have to move the decimal point 28 places to the left from its present position. We would thus have to place 27 zeros to the left of the 9.

Example 4–3. If the sun is $9.3 \cdot 10^7$ miles from the earth, how long does it take light to reach the earth from the sun?

Solution: As given in Example 4–1, the speed of light is $1.86 \cdot 10^5$ miles per second. Therefore, the required time is $\dfrac{9.3 \cdot 10^7}{1.86 \cdot 10^5} = 5 \cdot 10^2 = 500$ seconds = 8 minutes 20 seconds.

4–5. RATIONAL EXPONENTS

We shall now extend the meaning of exponents from integers to rational numbers. Here again we shall make the extension in such

a way that the laws for positive integral exponents will be preserved.

Suppose that a is a real number and that n is a positive integer. Let us assume that $a^{1/n}$ has meaning and that (4–4) applies. Then it would be true that

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a. \tag{4–10}$$

This says that the nth power of $a^{1/n}$ would have to be a, or in other words that $a^{1/n}$ would be what is called an nth *root* of a. For example, (4–10) would yield

$$(a^{1/2})^2 = a, \quad \text{and} \quad (a^{1/3})^3 = a.$$

Real nth Roots of a. Before defining $a^{1/n}$, let us examine the situation with respect to the existence of nth roots of a given number a. The following results may be proved with the help of the theory of equations.

Case I. If n is an even integer and a is a positive real number, there are two real numbers that satisfy the equation $r^n = a$. One of these is the positive nth root of a, which is denoted by $\sqrt[n]{a}$. The other is the negative nth root of a, which is denoted by $-\sqrt[n]{a}$. We may also denote these two numbers together by $\pm\sqrt[n]{a}$.

Case II. If n is an even integer and a is a negative real number, no real nth roots exist, since no even power of a real number can be negative.

Case III. If n is an odd integer and a is a positive real number, there is one real (positive) value of r such that $r^n = a$. In other words, if n is odd, there is a real positive nth root, which is denoted by $\sqrt[n]{a}$.

Case IV. If n is an odd integer and a is a negative real number, there exists one real (negative) value of r such that $r^n = a$. That is, if n is odd, there is a real negative nth root, which is denoted by $\sqrt[n]{a}$.

Case V. If n is any positive integer and a is zero, there is only one real nth root, and this root is zero.

Thus, the definition of an nth root of a is valid under all conditions except when a is negative and n is even. In this situation, no real nth roots exist. (However, the introduction of complex numbers in Chapter 11 will allow us to eliminate this exception.) We are now ready for the following definition.

Definition. If a is a non-negative real number and n is a positive integer, $a^{1/n}$ designates the non-negative nth root of a, or $\sqrt[n]{a}$. If a is negative and n is an odd positive integer, then $a^{1/n}$ designates the real nth root of a, or $\sqrt[n]{a}$.

When a is negative and n is even, $a^{1/n}$ is undefined.

Meaning of $a^{m/n}$. Let a be a given real number, n a positive integer, and m an integer. If $a^{m/n}$ has meaning, and if (4–4) holds for fractional powers, then $a^{m/n} = a^{(1/n)m} = (a^{1/n})^m$. Under these assumptions, then, $a^{m/n}$ would be the mth power of $a^{1/n}$. It is natural to state the following definition.

Definition. If n is a positive integer, if m is any integer such that the fraction m/n is in lowest terms, and if a is a real number which is assumed to be non-negative when n is even, then $a^{m/n}$ designates the mth power of $a^{1/n}$, that is, the mth power of $\sqrt[n]{a}$. Hence,

$$a^{m/n} = (a^{1/n})^m. \tag{4–11}$$

If the fraction m/n is not in its lowest terms, it is first reduced to lowest terms, and (4–11) is then applied.

When a is given, the value of $a^{m/n}$ depends only on the value of the fractional exponent, not on the particular values of m and n. Thus,

$$2^{4/2} = 2^2 = 4,\ 2^{6/8} = 2^{3/4}, \quad \text{and} \quad (-2)^{2/6} = (-2)^{1/3} = \sqrt[3]{-2}.$$

In the last example it would be incorrect to apply (4–11) directly, since $(-2)^{1/6}$ has no meaning.

It may be shown that, if a is positive,

$$a^{m/n} = \sqrt[n]{a^m}. \tag{4–12}$$

The proof is omitted.

We shall also omit the details of the procedure for showing that the five laws of exponents hold for rational exponents and non-negative bases. The reader is cautioned against using the laws for negative bases, since some fail under certain conditions.

To summarize the results now established, we restate the laws of exponents here for easy reference. It is assumed that a and b are non-negative real numbers, and that m and n are rational numbers. Furthermore, if either a or b appears in a denominator or raised to a negative or zero power, it is assumed to be different from zero.

Law of multiplication: $a^m \cdot a^n = a^{m+n}$.

Law of division: $\dfrac{a^m}{a^n} = a^{m-n}$.

Law for a power of a power: $(a^m)^n = a^{mn}$.

Law for a power of a product: $(ab)^n = a^n b^n$.

Law for a power of a quotient: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}\cdot$

Law for reciprocal: $a^{-n} = \dfrac{1}{a^n}\cdot$

Zero power: $a^0 = 1$.

Note that the radical notation can be replaced by the simpler and much more convenient exponential form. Everything that can be done with the radical notation in the simplification of roots of numbers and in operations involving roots can be done much more naturally by means of the exponential notation. A few illustrations of the meanings and uses of exponential forms follow:

$$x^{1/2} = \sqrt{x}, \text{ if } x \geqq 0; \quad (-27)^{1/3} = \sqrt[3]{-27} = -3;$$

$$-(32)^{1/5} = -\sqrt[5]{32} = -2; \quad -\sqrt[3]{a^6} = -a^2;$$

$$32^{2/5} = (\sqrt[5]{32})^2 = 2^2 = 4; \quad -(16x^4)^{-1/2} = \frac{-1}{(16x^4)^{1/2}} = \frac{-1}{\sqrt{16x^4}} = \frac{-1}{4x^2}.$$

The following examples illustrate the applications of the laws of exponents to the solution of problems involving radicals.

Example 4–4. Compute the value of $\sqrt[3]{2} \cdot \sqrt[4]{2}$, and write the result in exponential form.

Solution: Using exponential notation and the laws of exponents, we have

$$2^{1/3} \cdot 2^{1/4} = 2^{4/12} \cdot 2^{3/12} = 2^{4/12+3/12} = 2^{7/12}.$$

Example 4–5. Remove all possible factors from the radical $\sqrt[3]{162x^4y^2}$.

Solution: We may proceed as follows:

$$\sqrt[3]{162x^4y^2} = (2 \cdot 3^4x^4y^2)^{1/3} = 2^{1/3} \cdot 3^{4/3}\, x^{4/3}y^{2/3}$$

$$= 3x \cdot 2^{1/3}3^{1/3}x^{1/3}y^{2/3} = 3x\, \sqrt[3]{6xy^2}.$$

Example 4–6. Use the laws of exponents to express $\sqrt{x} \cdot \sqrt[3]{y}$ by using only one radical.

Solution: Changing to fractional exponents, we have

$$\sqrt{x} \cdot \sqrt[3]{y} = x^{1/2}y^{1/3} = x^{3/6}y^{2/6} = (x^3y^2)^{1/6} = \sqrt[6]{x^3y^2}.$$

Example 4–7. Rationalize the denominator in the fraction $\frac{1}{\sqrt[5]{x^3}}$.

Solution: To write an equivalent fraction in which no radical appears in the denominator, we proceed as follows:

$$\frac{1}{\sqrt[5]{x^3}} = \frac{1}{x^{3/5}} = \frac{1}{x^{3/5}} \cdot \frac{x^{2/5}}{x^{2/5}} = \frac{x^{2/5}}{x} = \frac{\sqrt[5]{x^2}}{x}.$$

Example 4–8. Rationalize the denominator of the fraction $\frac{1}{5 - \sqrt{3}}$.

Solution: We use the relationship $(a + b)(a - b) = a^2 - b^2$ to remove the radical from the denominator. Thus,

$$\frac{1}{5 - \sqrt{3}} = \frac{1}{5 - \sqrt{3}} \cdot \frac{5 + \sqrt{3}}{5 + \sqrt{3}} = \frac{5 + \sqrt{3}}{25 - 3} = \frac{5 + \sqrt{3}}{22}.$$

Example 4–9. Change $\dfrac{\sqrt{a^2 - x^2} + \dfrac{x^2}{\sqrt{a^2 - x^2}}}{a^2 - x^2}$ to a simple fraction.

Solution: Write the expression in exponential form, as follows:

$$\frac{(a^2 - x^2)^{1/2} + \dfrac{x^2}{(a^2 - x^2)^{1/2}}}{a^2 - x^2}.$$

Then, multiplying the main numerator and the main denominator by $(a^2 - x^2)^{1/2}$, we have

$$\frac{a^2 - x^2 + x^2}{(a^2 - x^2)^{3/2}} = \frac{a^2}{(a^2 - x^2)^{3/2}}.$$

Example 4–10. Express $(x^2 + a^2)^{3/2} + 3x^2(x^2 + a^2)^{1/2}$ in a factored form.

Solution: Rewrite the expression as

$$(x^2 + a^2)^{1/2}\,(x^2 + a^2) + 3x^2\,(x^2 + a^2)^{1/2}.$$

Removing the common factor $(x^2 + a^2)^{1/2}$, we obtain

$$(x^2 + a^2)^{1/2}\,[x^2 + a^2 + 3x^2] = (x^2 + a^2)^{1/2}\,(4x^2 + a^2).$$

EXERCISE 4–1

In each of the problems from 1 to 20, perform the indicated operations and eliminate all zero and negative exponents.

1. $3x^0y$.
2. $x^{-1/2}$.
3. $\left(\dfrac{3}{4}\right)^{-3}$.
4. 10^{-2}.
5. $\dfrac{1}{10^{-2}}$.
6. $\left(\dfrac{3}{8}\right)^0$.
7. $(9^6)^{1/3}$.
8. $5x^2y^0z^{-1}$.
9. $(x^3)^{-1/4}$.
10. $\dfrac{(xy^{1/3})^0}{(x^{1/4}y)^4}$.
11. $\dfrac{a^3y^{-7}}{a^2y^{-9}}$.
12. $3x^2y^{1/2}z^3 \cdot 4x^{1/2}yz^4$.
13. $(x^2y^2)^4(x^{-4}y^{-4})^2$.
14. $(x^{1/2}y^{-2})^4(x^2y^{-1/2})^0$.
15. $(x^{1/2} + y^{1/2})^2$.
16. $x^{-1} + y^{-1}$.
17. $(x + y)^{-1}$.
18. $(x + y)^{-1/3}(x + y)^{1/3}$.
19. $\dfrac{1 - x^{-1/2}}{1 + x^{-1/2}}$.
20. $\dfrac{x^{1/2} + x^{-1/2}}{x^{1/2} - x^{-1/2}}$.

Write each of the following expressions in exponential form. Remove all possible factors from the radical and, wherever necessary, rationalize the denominator.

21. $\sqrt{80}$.
22. $\sqrt[3]{-54}$.
23. $\sqrt[3]{21}$.
24. $\sqrt{xy^3}$.
25. $\sqrt[3]{x^2y^{-1}}$.
26. $\sqrt[4]{x^{-3}}$.
27. $\sqrt[4]{(2x^{-1})^3}$.
28. $\sqrt[3]{-x^4}$.
29. $\sqrt[3]{(-64)^0y^{-2}}$.
30. $\sqrt[5]{(4x^0y^{-3})^3}$.
31. $\sqrt{(81y^{-2})^{-1}}$.
32. $\sqrt{(a^2 + b^2)^{-1}}$.
33. $\sqrt{3} \cdot \sqrt[3]{14}$.
34. $\sqrt[3]{x^2} \cdot \sqrt{y}$.
35. $\sqrt{xy}\ \sqrt[3]{x^2y}$.
36. $\sqrt[6]{a^2b^3}\ \sqrt[4]{8a^3b^2}$.
37. $\dfrac{3}{3 - \sqrt{2}}$.
38. $\dfrac{1}{\sqrt{6} + 2}$.
39. $\dfrac{2}{1 - \sqrt{5}}$.
40. $\dfrac{1}{\sqrt{3} - \sqrt{2}}$.
41. $\dfrac{1}{x - \sqrt{x^2 - 9}}$.
42. $\dfrac{x + \sqrt{x^2 - 9}}{x - \sqrt{x^2 - 9}}$.
43. $\dfrac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}$.
44. $\dfrac{\sqrt{x} - \sqrt{x + 1}}{\sqrt{x} + \sqrt{x + 1}}$.

45. $\sqrt{2x - x^2} + \dfrac{x(1 - x)}{\sqrt{2x - x^2}}$.

46. $\dfrac{\sqrt{1 - x^2} + \dfrac{x^2}{\sqrt{1 - x^2}}}{1 - x^2}$.

47. $\dfrac{1 + \sqrt{1 + x^2} - \dfrac{x^2}{\sqrt{1 + x^2}}}{(1 + \sqrt{1 + x^2})^2}$.

48. $(x^2 + 1)^{3/2} + x^2(x^2 + 1)^{1/2}$.

49. $(2 - x^2)^{5/2} + x^2(2 - x^2)^{3/2}$.

50. $(x^2 - 3)^{1/2} - x^2(x^2 - 3)^{-1/2}$.

4-6. THE FACTORIAL SYMBOL

The product of all positive integers from 1 to n inclusive is called "n factorial" or "factorial n" and is represented by either of the symbols $n!$ or $\angle n$. Thus, if n is a positive integer,

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n - 1) \cdot n.$$

For example,

$$3! = 1 \cdot 2 \cdot 3 = 6; \quad 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120;$$

$$6! = 5! \cdot 6; \quad \frac{8!}{8} = 7!; \quad r! = [(r - 1)!]r.$$

4-7. THE BINOMIAL THEOREM

The statement known as the binomial theorem enables us to express any power of a binomial as a sum of terms without performing the multiplications.

By actually performing the indicated multiplications, we find that

$$\begin{aligned}
(a + b)^2 &= a^2 + 2ab + b^2,\\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3,\\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.
\end{aligned}$$

These formulas may be rewritten in the following manner, so as to suggest a general rule[1]:

$$\begin{aligned}
(a + b)^2 &= a^2 + \frac{2}{1}ab + \frac{2 \cdot 1}{1 \cdot 2}b^2,\\
(a + b)^3 &= a^3 + \frac{3}{1}a^2b + \frac{3 \cdot 2}{1 \cdot 2}ab^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}b^3,\\
(a + b)^4 &= a^4 + \frac{4}{1}a^3b + \frac{4 \cdot 3}{1 \cdot 2}a^2b^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}ab^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}b^4.
\end{aligned}$$

Applying this suggested rule to $(a + b)^5$, we obtain

$$\begin{aligned}
(a + b)^5 = a^5 &+ \frac{5}{1}a^4b + \frac{5 \cdot 4}{1 \cdot 2}a^3b^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}a^2b^3\\
&+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}ab^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}b^5.
\end{aligned}$$

[1] The justification for writing the expressions on the right in this form will be found in Chapter 17, where the binomial coefficients are given in terms of the *combination formulas*.

Upon simplification of coefficients, we get $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, which is the same result as that obtained by multiplying $(a+b)^4$ by $(a+b)$.

Each of the expressions on the left is of the form $(a+b)^n$, in which the exponents 2, 3, and 4 of $(a+b)$ are special values of n. If we let n denote the exponent of $(a+b)$ in each of the expressions on the left, we note that the expansion of $(a+b)^n$ contains $n+1$ terms with the following properties:

1. In any term the sum of the exponents of a and b is n. Also, the first term is a^n and the last term is b^n.
2. The exponent of a decreases by 1, and the exponent of b increases by 1, from term to term.
3. The denominator of the coefficient in each term is the factorial of the exponent of b in that term.
4. The numerator of the coefficient in each term has the same number of factors as the denominator. Specifically, wherever 1 appears in the denominator, write n directly above it in the numerator; wherever 2 appears in the denominator, write $n-1$ directly above it in the numerator; and so on. Thus, in $(a+b)^5$, the number above 1 is 5, and the number above 2 is 4.

Assuming that these properties hold for all positive integral values of n, we have

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n. \tag{4–13}$$

This result is the *binomial formula.* So far we have verified this formula only for $n = 2, 3, 4,$ and 5. In Chapter 16, we shall prove the *binomial theorem,* which states that the formula is true for all positive integral values of n.

The following example shows the procedure for the expansion of $(a+b)^n$.

Example 4–11. Expand $(x-2y)^6$ by the binomial theorem.

Solution: The required expansion will be obtained by letting $a = x$, $b = -2y$, and $n = 6$. We begin by setting up the following pattern of $n+1$ terms:

$$x^6 + \underline{\qquad}\, x^5(-2y) + \underline{\qquad}\, x^4(-2y)^2 + \underline{\qquad}\, x^3(-2y)^3 + \underline{\qquad}\, x^2(-2y)^4 + \underline{\qquad}\, x(-2y)^5 + (-2y)^6.$$

The exponents of b in the second, third, fourth, and fifth terms are 1, 2, 3, 4, and 5, respectively. Hence, remembering that the last term is b^n, we may fill in the numerators and denominators of the coefficients as follows:

$$x^6 + \frac{6}{1}x^5(-2y) + \frac{6\cdot 5}{1\cdot 2}x^4(-2y)^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}x^3(-2y)^3 + \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}x^2(-2y)^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}x(-2y)^5 + (-2y)^6.$$

By simplifying, we obtain the following result:

$$(x-2y)^6 = x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6.$$

4–8. GENERAL TERM IN THE BINOMIAL EXPANSION

If we wish to write any particular term of the expansion of $(a+b)^n$ without considering any of the other terms, a study of the binomial formula in (4–13) Section 4–7 will reveal the following facts:

In every term the exponent of b is one less than the number of the term. Thus, in the $(r+1)$th term, the exponent of b is r. (The expression for a particular term is simplified slightly if the number of that term is called $r+1$, rather than r.)

The sum of the exponents of a and b is n in each term. For the $(r+1)$th term the exponent of a is $n-r$.

The denominator of the coefficient in the $(r+1)$th term is $r!$, since it is the factorial of the exponent of b.

The numerator of the coefficient has the same number of factors as the denominator. In the $(r+1)$th term, it is the product $n(n-1)(n-2)\cdots(n-r+1)$.

We obtain, then, for the $(r+1)$th term of the expansion $(a+b)^n$,

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r.$$

Example 4–12. Find the sixth term of $(3x-y^2)^8$.

Solution: Here $a = 3x$, $b = -y^2$, and $n = 8$. Since $r+1 = 6$, the exponent of b is $r = 5$. Hence, the exponent of a is $n - 5 = 3$. Therefore, the sixth term is

$$\frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5}(3x)^3(-y^2)^5 = -1512x^3y^{10}.$$

EXERCISE 4–2

In each of the problems from 1 to 16, reduce the given fraction to lowest terms.

1. $\dfrac{(9!)(3!)}{8!}$.
2. $\dfrac{10!}{(3!)(7!)}$.
3. $\dfrac{(1!)(3!)}{(2!)(2!)}$.
4. $\dfrac{(7!)(8!)}{(5!)(6!)}$.
5. $\dfrac{(3\cdot 4)!}{3(4!)}$.
6. $\dfrac{(3\cdot 4)!}{(3!)(4!)}$.
7. $\dfrac{3!+4!}{(3!)(4!)}$.
8. $\dfrac{n!}{(n-1)!}$.
9. $\dfrac{n!}{(n-2)!}$.
10. $\dfrac{n!}{(n-r)!}$.
11. $\dfrac{(n+1)!}{(n-1)!}$.
12. $\dfrac{n!}{3!\,(n-3)!}$.
13. $\dfrac{(n+1)!\,(n-1)!}{(n!)^2}$.
14. $\dfrac{[(n+1)!]^2}{n!\,(n-2)!}$.
15. $\dfrac{n!\,(n+1)!}{(n-1)!\,(n+2)!}$.
16. $\dfrac{n!\,(n-2)!}{[(n-1)!]^2}$.

17. Show that $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \dfrac{n!}{r!\,(n-r)!}$.

In each of the problems from 18 to 32, expand the given expression by the binomial theorem. (Hint: In problems 29 thru 32, first consider the first two terms in parentheses as a single quantity).

18. $(x+y)^5$.
19. $(x-1)^7$.
20. $(a-2b)^5$.
21. $(2a^2-3b^2)^3$.
22. $(\sqrt{x}+\sqrt{y})^6$.
23. $\left(\sqrt{x}+\dfrac{2}{\sqrt{x}}\right)^5$.
24. $\left(\dfrac{x}{2}-\dfrac{2}{x}\right)^5$.
25. $\left(\dfrac{x^2}{y}-\dfrac{y}{x^2}\right)^4$.
26. $(-x+y^{-2})^4$.
27. $(x^2-y^2)^6$.
28. $\left(\dfrac{y^2}{x}+\dfrac{a^{1/3}}{y}\right)^5$.
29. $(x+y+z)^2$.
30. $(x+2y+z)^2$.
31. $(x^2+x+1)^4$.
32. $(a^2-a-1)^3$.

In each of the problems from 33 to 42, find the indicated term.

33. $(1-x)^8$, 8th term.
34. $(m+n)^{14}$, 10th term.
35. $(a+2b)^{12}$, 5th term.
36. $(a-b)^{11}$, 3rd term.
37. $\left(\dfrac{x}{2}-\dfrac{2}{x}\right)^8$, 4th term.
38. $(x-y)^{10}$, term involving y^4.
39. $(2x^3+3y^2)^{15}$, term involving y^8.
40. $\left(\dfrac{x}{y}-\dfrac{y^2}{x^2}\right)^{11}$, term involving $\dfrac{1}{y^2}$.
41. $(\sqrt{x}-\sqrt{y})^{12}$, middle term.
42. $(2x-y)^7$, middle terms.

5 Logarithms

5–1. DEFINITION OF A LOGARITHM

We shall assume here that the laws of exponents stated in Chapter 4 for rational exponents are valid also for irrational exponents. The definition of a base raised to an irrational power is beyond the scope of this book. However, let us make the assumption that, if b and x are real numbers, with b positive, a corresponding number designated by b^x exists. Without giving an explicit rule for computing b^x, let us assume that all laws of exponents established in Chapter 4 are valid generally for real powers. Finally, let us assume that, corresponding to any two positive real numbers b and n, where $b \neq 1$, there exists a unique real number x, such that $n = b^x$. We can then give the following definition.

Definition. If $n = b^x$, where b is a positive real number different from 1, then x is called the *logarithm of n to the base b*. We write $x = \log_b n$. The following table shows both forms of several equivalent statements.

Exponential Form	$2^3 = 8$	$4^{1/2} = 2$	$3^{-4} = \frac{1}{81}$	$5^0 = 1$
Logarithmic Form	$\log_2 8 = 3$	$\log_4 2 = \frac{1}{2}$	$\log_3 \frac{1}{81} = -4$	$\log_5 1 = 0$

We shall restrict n to positive numbers, since negative numbers do not have real logarithms.

For any positive base b, we have $b^0 = 1$ and $b^1 = b$. Hence, it follows from the definition of a logarithm that

$$\log_b 1 = 0 \quad \text{and} \quad \log_b b = 1.$$

Another valuable relationship results from combining the two equations $n = b^x$ and $x = \log_b n$. Replacing x in the first equation by its value from the second equation, we have

$$n = b^{\log_b n}.$$

For example, $2^{\log_2 8} = 8$, and $10^{\log_{10} x} = x$.

Example 5–1. Find n, if $\log_3 n = 2$.

Solution: Write the given equation in exponential form, as follows:

$$n = 3^2.$$

Hence, $n = 9$.

Example 5–2. Find the base b, if $\log_b 4 = 2/3$.

Solution: In exponential form, the given equation is $b^{2/3} = 4$. Raise both sides to the 3/2 power and recall that $b > 0$. Then

$$(b^{2/3})^{3/2} = b = 4^{3/2}.$$

Therefore, $b = 8$.

Example 5–3. Find x, if $\log_{1/8} 32 = x$.

Solution: Writing the equation in exponential form, we have

$$\left(\frac{1}{8}\right)^x = 32.$$

Express 1/8 and 32 as powers of 2, and get $1/8 = 1/2^3 = 2^{-3}$, and $32 = 2^5$. Hence,

$$(2^{-3})^x = 2^5, \quad \text{or} \quad 2^{-3x} = 2^5.$$

From this, we have $-3x = 5$, and $x = -5/3$. Therefore, $\log_{1/8} 32 = -5/3$.

5–2. LAWS OF LOGARITHMS

Since a logarithm is an exponent with respect to a given base, the rules for operating with logarithms are the same as the laws of exponents. These laws, expressed in terms of logarithms, have the following form.

Law I. The logarithm of a product equals the sum of the logarithms of its factors. The logarithmic form is

$$\log_b (m \cdot n) = \log_b m + \log_b n. \tag{5–1}$$

Proof: To prove this equation, let

$$x = \log_b m \quad \text{and} \quad y = \log_b n.$$

Then

$$m = b^x \quad \text{and} \quad n = b^y.$$

Multiplying, we have

$$mn = b^{x+y}.$$

Hence, .

$$\log_b (mn) = x + y = \log_b m + \log_b n.$$

Law II. The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor. The logarithmic form is

$$\log_b \left(\frac{m}{n}\right) = \log_b m - \log_b n. \tag{5–2}$$

Proof: The proof follows: Let

$$x = \log_b m \quad \text{and} \quad y = \log_b n.$$

Then

$$m = b^x \text{ and } n = b^y.$$

Dividing, we have

$$\frac{m}{n} = b^{x-y}.$$

Hence,

$$\log_b \left(\frac{m}{n}\right) = x - y = \log_b m - \log_b n.$$

Law III. The logarithm of a power of a number equals the exponent times the logarithm of the number; that is,

$$\log_b (n^k) = k \log_b n. \tag{5-3}$$

Proof: The first step in the proof is to let

$$x = \log_b n.$$

Then

$$n = b^x.$$

Raise both sides to the kth power and obtain

$$n^k = (b^x)^k = b^{kx}.$$

This relationship, when written in logarithmic form, becomes

$$\log_b (n^k) = kx.$$

Replacing x by its value, we have

$$\log_b (n^k) = k \log_b n.$$

The student should note carefully the difference between $\log_b (n^k)$ and $(\log_b n)^k$.

Law IV. The logarithm of a root of a number equals the logarithm of the number divided by the index of the root; that is,

$$\log_b \sqrt[k]{n} = \frac{1}{k} \log_b n. \tag{5-4}$$

Proof: This equation follows as a corollary of law III. By the definition of a fractional exponent, we have $\sqrt[k]{n} = n^{1/k}$. Hence, by law III,

$$\log_b \sqrt[k]{n} = \log_b (n^{1/k}) = \frac{1}{k} \log_b n.$$

Example 5–4. Express $\log_2 \dfrac{\sqrt{51}}{3^4}$ as a linear combination of logarithms.

Solution: $\log_2 \dfrac{\sqrt{51}}{3^4} = \log_2 \sqrt{51} - \log_2 3^4 = \log_2 (3 \cdot 17)^{1/2} - \log_2 3^4$

$$= \log_2 3^{1/2} + \log_2 17^{1/2} - \log_2 3^4.$$

Example 5-5. Express $2 \log_{10} 3 - \frac{1}{2} \log_{10} x + \log_{10} y$ as a single logarithm.

Solution:
$$
\begin{aligned}
2 \log_{10} 3 - \frac{1}{2} \log_{10} x + \log_{10} y &= \log_{10} 3^2 - \log_{10} x^{1/2} + \log_{10} y \\
&= \log_{10} (3^2 \cdot y) - \log_{10} x^{1/2} \\
&= \log_{10} \frac{3^2 y}{x^{1/2}} = \log_{10} \frac{9y}{x^{1/2}}.
\end{aligned}
$$

Example 5-6. Transform the equation $\log_a x + y = \log_a \sin x$ into an equation free of logarithms.

Solution: By transposing, we get

$$y = \log_a \sin x - \log_a x = \log_a \frac{\sin x}{x}.$$

Change to the following exponential form:

$$a^y = \frac{\sin x}{x}.$$

EXERCISE 5-1

In each of the problems from 1 to 12, write the equation in logarithmic form.

1. $2^3 = 8.$ **2.** $2^6 = 64.$ **3.** $3^4 = 81.$ **4.** $10^0 = 1.$

5. $10^3 = 1000.$ **6.** $10^{-3} = 0.001.$ **7.** $256^{1/8} = 2.$ **8.** $216^{1/3} = 6.$

9. $100^{0.5} = 10.$ **10.** $y = e^x.$ **11.** $10^y = x.$ **12.** $10^{\log y} = x.$

In each of the problems from 13 to 21, write the equation in exponential form.

13. $\log_8 64 = 2.$ **14.** $\log_5 125 = 3.$ **15.** $\log_2 \frac{1}{64} = -6.$

16. $\log_5 \frac{1}{625} = -4.$ **17.** $\log_7 343 = 3.$ **18.** $\log_9 729 = 3.$

19. $\log_{10} 10{,}000 = 4.$ **20.** $\log_{10} 0.0001 = -4.$ **21.** $\log_4 8 = 3/2.$

In each of the problems from 22 to 33, find the indicated value of x.

22. $\log_9 3 = x.$ **23.** $\log_2 64 = x.$ **24.** $\log_4 x = 0.$

25. $\log_x 4 = 2.$ **26.** $\log_{0.5} x = -1.$ **27.** $\log_3 x = 1.$

28. $\log_x 81 = 4.$ **29.** $\log_x 100 = -2.$ **30.** $\log_x \frac{1}{32} = 5.$

31. $\log_9 243 = x.$ **32.** $\log_{64} x = -\frac{7}{6}.$ **33.** $\log_x x = 2.$

In each of the problems from 34 to 39, use the laws of logarithms to write the expression as a single logarithm.

34. $\log_b 2 - 3 \log_b 5 + \log_b 7.$ **35.** $\log_b 4 + \log_b \pi - \log_b 3 + 3 \log_b r.$

36. $\frac{1}{5} \log_b 7 + \frac{2}{5} \log_b 4 + \frac{1}{5} \log_b 3.$ **37.** $-5 \log_b 23 + 12 \log_b \frac{23}{2}.$

38. $3 \log_b 2 + \log_b 13 - 2 \log_b 5.$

39. $\frac{1}{2}\log_b(u - \sqrt{u^2 - a^2}) - \frac{1}{2}\log_b(u + \sqrt{u^2 - a^2}) + \log_b a.$

40. Find the logarithm to the base b of the area of a circle in terms of the logarithms of π and the radius.

41. The time T for a pendulum of length l to make one oscillation is $T = \pi\sqrt{\frac{l}{g}}$, where g is a constant representing the acceleration due to gravity. *a*) Find $\log_b T$ in terms of the logarithms of π, l, and g. *b*) Find $\log_b l$ in terms of the logarithms of π, T, and g.

42. The area of a triangle with sides of length a, b, and c is given by the formula $K = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter $\frac{1}{2}(a + b + c)$. Find $\log_b K$ in terms of the logarithms of combinations of a, b, and c.

43. The positive geometric mean G of n positive numbers $x_1, x_2, \cdots . x_n$ is defined by the relationship

$$\log_b G = \frac{\log_b x_1 + \log_b x_2 + \cdots + \log_b x_n}{n}.$$

Show that $G = \sqrt[n]{x_1 x_2 \cdots x_n}$.

5–3. SYSTEMS OF LOGARITHMS

As we mentioned in Section 5–1, any positive number b different from 1 may be used as a base in a system of logarithms. However, only two bases are widely used in practice.

The *common,* or *Briggs,* system of logarithms, named for Henry Briggs (1556-1631), employs the base 10 and is used for ordinary computations.

The *natural,* or *Napierian,* system of logarithms, named for John Napier (1550-1617), is generally used in calculus and theoretical work, and employs the more convenient irrational base $e = 2.71828 \cdots$.

In this book, when the base is not indicated, it is understood to be 10. Thus, $\log n$ means $\log_{10} n$, and the word *logarithm* will mean *common logarithm* unless otherwise stated.

5–4. COMMON LOGARITHMS

In Table 5–1, we begin with a list of powers of 10, give equivalent logarithmic forms, and from these determine the form of the logarithm of a number that is not an exact power of 10. It should be mentioned that the logarithm is an increasing function; that is, as n increases, $\log n$ increases. Another way of stating the conditions is to say that if $a > b$ then $\log a > \log b$.

TABLE 5–1

Exponential form	Logarithmic form	Logarithm of the number
$10^3 = 1000$	$\log 1000 = 3.000$	
		$\leftarrow \log 354. = 2 + \text{decimal}$
$10^2 = 100$	$\log 100 = 2.000$	
		$\leftarrow \log 35.4 = 1 + \text{decimal}$
$10^1 = 10$	$\log 10 = 1.000$	
		$\leftarrow \log 3.54 = 0 + \text{decimal}$
$10^0 = 1$	$\log 1 = 0.000$	
		$\leftarrow \log 0.354 = -1 + \text{decimal}$
$10^{-1} = 0.1$	$\log 0.1 = -1.000$	
		$\leftarrow \log 0.0354 = -2 + \text{decimal}$
$10^{-2} = 0.01$	$\log 0.01 = -2.000$	
		$\leftarrow \log 0.00354 = -3 + \text{decimal}$
$10^{-3} = 0.001$	$\log 0.001 = -3.000$	

From Table 5–1 it can be seen that the following statements are true:

The logarithm of an integral power of 10 is an integer.

The logarithm of a number which is not an integral power of 10 consists of two terms or parts: an integral part, called the *characteristic;* and a positive or zero decimal part, called the *mantissa,* which is determined from a table of mantissas.

Thus, since $\log 10 = 1$ and $\log 100 = 2$, we may expect the logarithm of any number between 10 and 100, that is, a number between 10^1 and 10^2, to be 1 plus a positive decimal part. For example, we shall find that the logarithm of 35.4, which number lies between 10 and 100, is equal to 1.5490, to four decimal places. In this case, the characteristic is 1 and the mantissa is .5490.

5–5. RULES FOR CHARACTERISTIC AND MANTISSA

A study of Table 5–1 reveals that the characteristic changes as the position of the decimal point changes in the sequence of digits 0035400. The first entry in the column headed "Logarithm of the number" is

$$\log 354 = 2. + \text{decimal}.$$

In the number 354, or 354.0, the decimal point is *two* places to the right of the first non-zero digit, 3 (reading from left to right); the corresponding characteristic is 2.

The second entry is

$$\log 35.4 = 1. + \text{decimal}.$$

In this number, 35.4, the decimal point is *one* place to the right of the first non-zero digit (reading from left to right); the corresponding characteristic is 1.

Similarly, we note that the *zero characteristic* corresponds to the position of the decimal point immediately following the first non-zero digit. This position of the decimal point is called the *standard position.* We may now formulate the following rule for characteristics:

Rule for Characteristics. If the decimal point is in standard position, the characteristic is zero. For every other position of the decimal point, the characteristic is equal to the number of places the decimal point has been shifted from the standard position. The characteristic is positive if the shift is to the right, and is negative if the shift is to the left.

We shall now see that the mantissa remains the same for all numbers having the same sequence of digits. Let us again consider the sequence of digits 0035400. Any number containing this sequence can be written $3.54 \cdot 10^n$, where n is a positive or negative integer or zero and depends on the position of the decimal point. Suppose that we consider the form $\log 3.54 = 0.5490$. Then the logarithm of any number containing this sequence is

$$\begin{aligned}\log (3.54 \cdot 10^n) &= \log 3.54 + \log 10^n\\ &= n + \log 3.54\\ &= n + 0.5490.\end{aligned}$$

Thus, a shift of the decimal place in the number affects only the characteristic n, and the mantissa remains the same for the same sequence of digits.

EXERCISE 5–2

In each of the problems from 1 to 16, find the characteristic of the logarithm of the given number.

1. 34.63.	**2.** 3.463.	**3.** 34630.	**4.** 268.1.
5. 0.1340.	**6.** 2637.	**7.** 0.00346.	**8.** tan 42°8′.
9. sin 63°41′.	**10.** 0.000001.	**11.** 378364.	**12.** $\dfrac{7.821}{10{,}000}$.
13. cot 81°13′.	**14.** sin 84°53′.	**15.** cos 61°43′.	**16.** sec 24°8′.

In each of the problems from 17 to 24, place the decimal point in the sequence of digits 7314 corresponding to the given characteristic.

17. 3.	**18.** -2.	**19.** 0.	**20.** 1.
21. 6.	**22.** -5.	**23.** -3.	**24.** -1.

5–6. HOW TO WRITE LOGARITHMS

As stated in Section 5–4, the mantissa of a logarithm is always positive or zero, whereas the characteristic may be a positive or negative integer or zero. A positive characteristic or a zero characteristic can readily be combined with a given mantissa. For example, the logarithm of 354 is written 2.5490. But when the characteristic is negative, say $-k$, where $1 \leqq k \leqq 10$, it is more convenient to write it in the form $(10 - k) - 10$. Let us consider the logarithm of 0.00354. The characteristic is -3, but the mantissa is regarded as positive. We could write $\log 0.00354 = -3 + 0.5490$. For convenience in computation, however, we write log 0.00354 in the form $(10 - 3) + 0.5490 - 10 = 7.5490 - 10$, or $17.5490 - 20$, and so on.

Note. It would be incorrect to write $\log 0.00354 = -3.5490$, for this notation means $-3 - 0.5490$ and would imply that the mantissa is negative. To perform certain computations, it is convenient to write the logarithm $7.5490 - 10$ in the form -2.4510, which equals $-2 - 0.4510$. It is important to note that the decimal part of the number -2.4510 is not the mantissa of the logarithm of 0.00354, since it is not positive.

5–7. HOW TO USE A TABLE OF MANTISSAS

The following examples will illustrate the procedure in finding the logarithm of a number with the aid of a table of mantissas. The student should work through each example, determining the characteristic from the position of the decimal point in the number and determining the mantissa by referring to Table III at the end of this book.

Example 5–7. Find log 46.7.

Solution: The characteristic is $+1$. To find the mantissa, locate 46 in the column in the table headed N, and then go to the right to the column headed 7. Here we find the mantissa .6693. So the complete result is $\log 46.7 = 1.6693$.

Interpolation. If the number consists of more than three digits, the mantissa is found from Table III by means of interpolation. Since the method of interpolation is the same as that described in Section 3–10 for the table of trigonometric functions, there will be no further discussion of it here.

Example 5–8. Find log 0.03426.

Solution: The characteristic is -2. The mantissa is found by interpolation, since the number 3426 has more than three digits. It lies $\frac{6}{10}$ of the way between the mantissas of 3420 and 3430, as shown in the accompanying tabulation:

$$\begin{array}{cc} \text{Number} & \text{Mantissa} \end{array}$$

$$10\left\{\begin{array}{l} 6\left\{\begin{array}{ll} 3420 & .5340 \\ 3426 & .5340 + x \end{array}\right\} x \\ \begin{array}{ll} 3430 & .5353 \end{array} \end{array}\right\} 13$$

Since the difference between the mantissas of the two numbers in the table is 13, we have

$$x = \frac{6}{10}(13) = 7.8.$$

This is rounded off to 8, and the amount to be added to 0.5340 is given by $x = 8$. Hence, the mantissa is .5348 and log 0.03426 = 8.5348 − 10.

Finding Antilogarithms. The number which corresponds to a given logarithm is called the *antilogarithm*. That is, if $\log n = x$, then n is the antilogarithm of x and is written antilog x.

Example 5–9. Find n, if $\log n = 1.8710$.

Solution: Search through the body of Table III to locate the mantissa .8710. The corresponding number, from the columns headed N and 3, is 743. Since the characteristic is 1, $n = 74.3$.

Example 5–10. Find antilog 7.5349–10.

Solution: The mantissa .5349 is not in Table III but lies between .5340 and .5353. To these correspond, respectively, numbers whose digits are 3420 and 3430. We may indicate the work in tabular form as follows:

$$\begin{array}{cc} \text{Number} & \text{Mantissa} \end{array}$$

$$10\left\{\begin{array}{l} x\left\{\begin{array}{ll} 3420 & .5340 \\ 3420 + x & .5349 \end{array}\right\} 9 \\ \begin{array}{ll} 3430 & .5353 \end{array} \end{array}\right\} 13$$

From this, we see that

$$\frac{x}{10} = \frac{9}{13}.$$

Therefore, $x = 6.9$, or 7 after rounding off. Hence, the sequence of digits in the desired number is 3427. Since the characteristic is -3, the antilogarithm of 7.5349–10 is 0.003427.

EXERCISE 5–3

In each of the problems from 1 to 30, find the common logarithm of the given number.

1. 35. **2.** 98. **3.** 105. **4.** 0.0843. **5.** 0.00621.

6. 9.63. **7.** 23,100. **8.** 12.3. **9.** 0.354. **10.** 0.0781.

11. 0.0663. **12.** 1,630. **13.** log 7.03. **14.** log 95.5. **15.** log 695.

16. log 6.31. **17.** 0.007001. **18.** 0.003821. **19.** 0.7777. **20.** 7,437.

21. 3.142. **22.** 1.414. **23.** 0.08788. **24.** 15.46. **25.** cos 16°13′.

26. sin 10°18′. **27.** tan 41°33′. **28.** sec 64°16′. **29.** cos 82°14′. **30.** cot 31°16′.

In each of the problems from 31 to 50, find the antilogarithm of the given number.

31. 1.6665. **32.** 4.4857. **33.** 9.4183–10. **34.** 0.0645.

35. 2.7024. **36.** 7.7388–10. **37.** 9.4409–20. **38.** 6.3404–10.

39. 1.8401. **40.** 3.9552–10. **41.** 2.4658. **42.** 1.9501.

43. 9.7367. **44.** 4.9960–10. **45.** 8.7863–10. **46.** 9.8821–20.

47. 0.6584. **48.** 3.0150. **49.** 5.0300–10. **50.** 0.1504.

Solve for x in each of the following equations:

51. $10^x = 4$. **52.** $10^x = 2.019$. **53.** $10^{2x} = 7.132$.

54. $10^{0.1} = x$. **55.** $\sqrt[5]{10} = x$. **56.** $10^{\sqrt{2}+1} = x$.

57. $\sqrt[7]{10^4} = x$. **58.** $10^{1.314} = x$. **59.** $10^{-x} = 0.003146$.

60. $10^{-x/2} = 0.0123$. **61.** $10^{1-x} = 0.2346$. **62.** $10^{2x-3} = 0.6735$.

5–8. LOGARITHMIC COMPUTATION

The fundamental laws of logarithms given in Section 5–2 are applied in the following examples to illustrate the application of logarithms to computation.

Example 5–11. Find the product $(0.0246) \cdot (1360)$.

Solution: Let $x = (0.0246) \cdot (1360)$. Then

$$\log x = \log 0.0246 + \log 1360.$$

$$\begin{aligned} \log 0.0246 &= 8.3909 - 10 \\ \log 1360 &= \underline{3.1335} \\ \log x &= 11.5244 - 10 \\ &= 1.5244. \end{aligned}$$

Hence, by interpolation, we have $x = 33.45$.

Example 5–12. Evaluate $(0.506)^{-1/3}$.

Solution: Let $x = (0.506)^{-1/3} = \dfrac{1}{(0.506)^{1/3}}$. Then

$$\begin{aligned}
\log x &= \log 1 - \log (0.506)^{1/3} \\
&= \log 1 - (1/3) \log 0.506 \\
&= \log 1 - (1/3)\ (29.7042\text{–}30) \\
&= \log 1 - (9.9014\text{–}10).
\end{aligned}$$

$$\begin{aligned}
\log 1 &= 10.0000\text{–}10 \\
1/3 \log 0.506 &= \underline{\ 9.9014\text{–}10} \\
\log x &= 0.0986.
\end{aligned}$$

Therefore, $x = 1.255$ by interpolation.

Alternate Solution: Let $x = (0.506)^{-1/3}$. Then

$$\begin{aligned}
\log x &= -\ (1/3) \log 0.506 \\
&= -\ (1/3)\ (29.7042\text{–}30) \\
&= -\ (9.9014\text{–}10) \\
&= -\ (-\ 0.0986) \\
&= 0.0986.
\end{aligned}$$

Therefore, $x = 1.255$ by interpolation.

Example 5–13. Evaluate $\dfrac{(0.352)\ (1.74)^2}{\sqrt[3]{0.00526}}$.

Solution: Let x denote the desired value. Then

$$\log x = \log 0.352 + 2 \log 1.74 - (1/3) \log 0.00526.$$

We find that $\log 0.352 = 9.5465\text{–}10$, $\log 1.74 = 0.2405$, and $\log 0.00526 = 7.7210\text{–}10$.

$$\begin{aligned}
\log 0.352 &= \ 9.5465\text{–}10 \\
2 \log 1.74 &= \underline{\ 0.4810\qquad} \\
\text{log numerator} &= 10.0275\text{–}10
\end{aligned}
\qquad
\begin{aligned}
(1/3) \log 0.00526 &= (1/3)\ (27.7210\text{–}30) \\
&= 9.2403\text{–}10.
\end{aligned}$$

$$\begin{aligned}
\text{log numerator} &= 10.0275\text{–}10 \\
\text{log denominator} &= \underline{\ 9.2403\text{–}10} \\
\log x &= \ 0.7872.
\end{aligned}$$

Interpolating, we have $x = 6.126$.

Example 5–14. Evaluate $\left(\dfrac{253}{174}\right)^{1.14}$

Solution: Let $x = \left(\dfrac{253}{174}\right)^{1.14}$. Then

$$\log x = 1.14\ [\log 253 - \log 174].$$

$$\begin{aligned}
\log 253 &= 2.4031 \\
\log 174 &= \underline{2.2405} \\
&\ 0.1626.
\end{aligned}$$

Then

$$\log x = 1.14\ (0.1626) = 0.1854.$$

Therefore, $x = 1.532$.

EXERCISE 5–4

In each of the problems from 1 to 30, perform the indicated computation using logarithms.

1. $(3.142)(2.718)$.

2. $(13.25)(26.80)$.

3. $\dfrac{29.34}{683.5}$.

4. $(0.8134)^{1/3}$.

5. $\sqrt[3]{(5.678)^2}$.

6. $(16.83)^{3/4}$.

7. $\sqrt[4]{(0.003468)^5}$.

8. $\dfrac{836.1}{42,860}$.

9. $\sqrt[5]{\dfrac{(5,321,000)^2}{(36,250)^4}}$.

10. $(4.313)(3,068)(0.000642)$.

11. $(63.84)^2(0.0134)$.

12. $(8.364)(321.5) \div (-42.63)$.

13. $\sqrt{(168.3)\ (14.21)}$.

14. $\sqrt{(213.6)^2(43.98)^2}$.

15. $\sqrt{(23,310)^2 - (20,180)^2}$. (Hint: Factor the radicand.)

16. $\left(\dfrac{68.34}{21.37}\right)^{4/5}$.

17. $\left(\dfrac{123.4}{567.8}\right)^{9/8}$.

18. $\dfrac{3,642\ \sqrt[4]{(21.36)^3}}{(1,083)^4(0.0813)^3}$.

19. $\dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 19 \cdot 31}{2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64}$.

20. $\dfrac{1}{38.63}\sqrt[3]{\dfrac{3.483}{0.8103}}$.

21. $\sqrt[5]{\dfrac{(31.63)^4}{(8.013)(0.034)}}$.

22. $(1.08)^{10}$.

23. $(18,120) \cdot \dfrac{(1.04)^{20} - 1}{0.04}$.

24. $3,648\ (1.03)^{35}$.

25. $\sqrt[4]{68.43} - (0.8123)^{-3/4}$.

26. $0.083^{0.412}$.

27. $\sqrt[3]{\log 0.08614}$.

28. $\log 16.84 - \sqrt[3]{483.6}$.

29. $[(3.864)^{-3.13} + (0.841)^{-0.68}]^{2/3}$.

30. $\dfrac{(83.14)^{-4.3} - (0.8134)^{2/3}}{(0.6841)^{1/2}}$.

31. If $e = 2.718$, find $\log e$, $\log \sqrt{e}$, $\log \dfrac{1}{e}$, e^{π}, and π^{e}.

32. Find the geometric mean of 564.3, 8634, 0.1349, 8.316, and 42.61. (Hint: See Problem 43, Exercise 5–1.)

33. Find the area of a circle of radius 6,381 feet.

34. The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$. Find the volume of a sphere of radius 3621 feet.

35. Find the radius of a sphere whose volume is 8423 cubic feet.

36. Find the length of a pendulum which makes one oscillation in 1 second, if $g = 980$ centimeters/sec². (Hint: See Problem 41, Exercise 5–1.)

37. Find the area of a triangle with sides 6,384 feet, 5,680 feet, and 2,164 feet long. (Hint: See Problem 42, Exercise 5–1.)

38. The stretch s of a wire of length l and radius r by a weight m is given by the relationship $s = \frac{mgl}{\pi r^2 k}$, where g is the gravitational constant and k (Young's modulus) is a constant for a given material. Find how much a copper wire of length 120 centimeters and of radius 0.040 centimeters will be stretched by a weight of 6,346 grams, if $g = 980$ and k is $1.2 \cdot 10^{12}$ for copper wire.

39. The current i flowing in a series circuit with a resistance of R ohms and L henrys t seconds after the source of electromotive force is short-circuited is given by the relationship $i = Ie^{-Rt/L}$, where I is the current flowing in the circuit before the short circuit. If $i = 10$ amperes, $R = 0.1$ ohms, $t = 0.25$ seconds, and $L = 0.05$ henrys, find I. (Take $e = 2.718$.)

40. If n is a positive integer, $n!$ has been defined as the product $1 \cdot 2 \cdot \cdots \cdot n$. When n is very large, it is difficult to compute this product. However, Stirling's formula gives (approximately) $n! = n^n e^{-n} \sqrt{2\pi n}$. Use this formula to estimate 9!, and compare the result with the true value which you should calculate exactly. Do the same with 30!. (Hint: $\log (n!) = n \log n - n \log e + \frac{1}{2} \log 2 + \frac{1}{2} \log \pi + \frac{1}{2} \log n$).

41. If the rate of depreciation r per year is constant, the scrap value S after n years of a machine with first cost C is given by the formula $S = C(1 - r)^n$. Find the scrap value after 10 years of a machine which originally cost \$10,000, if 20 per cent per year is written off as depreciation.

5–9. CHANGE OF BASE

It is sometimes desirable to change from one logarithmic base to another. Suppose there is available a table of logarithms to some known base b (say 10, for example), and we wish to find the logarithm of a number n to some other base a. We then let $x = \log_b n$; whence, by definition, $n = b^x$. Similarly, if we let $y = \log_a n$, then we have $n = a^y$.

It follows that $a^y = b^x$, and our problem reduces to solving this equation for y. Taking the logarithm of both sides to base b, we have

$$y \log_b a = x \log_b b.$$

But $\log_b b = 1$. Therefore,

$$y = x \left(\frac{1}{\log_b a}\right) = \log_b n \left(\frac{1}{\log_b a}\right)$$

or

$$\log_a n = \frac{\log_b n}{\log_b a}. \tag{5–5}$$

Example 5–15. Find $\log_e 125$, where $e = 2.7183$, by using a table to the base 10.

Solution: By (5–5),

$$\log_e 125 = \frac{\log_{10} 125}{\log_{10} e}$$

It is usually easier to multiply than to divide. Since division by $\log_{10} e$ is a fairly frequent operation in practical work, it should be noted that $\frac{1}{0.4343} = 2.3026$, and the result can be obtained by multiplying by 2.3026 instead of dividing by 0.4343. Thus,

$$\log_e 125 = (2.0969)\ (2.3026) = 4.828.$$

EXERCISE 5–5

Find each of the following logarithms by using a table of common logarithms:

1. $\log_e 10$. **2.** $\log_9 100$. **3.** $\log_2 e$. **4.** $\log_e \pi$.

5. $\log_\pi e$. **6.** $\log_\pi 10$. **7.** $\log_2 64$. **8.** $\log_{20} 1000$.

9. $\log_{20} 100$. **10.** $\log_{100} 64$. **11.** $\log_e 8$. **12.** $\log_{0.1} 50$.

13. $\log_{125} 1000$. **14.** $\log_e 20$. **15.** $\log_{0.02} 0.04$. **16.** $\log_{1000} 100$.

6 Right Triangles and Vectors

6–1. ROUNDING OFF NUMBERS

Numbers that arise in the applications of trigonometry are usually not exact, but are sufficiently accurate for a given purpose. Numbers of this kind are called *approximate numbers*, and the degree of accuracy of such a number is indicated by how many *significant figures* it contains. Reading from left to right, the significant figures in a number are the digits starting with the first non-zero digit and ending with the last non-zero digit, unless it is definitely specified that the zeros on the right are significant. Thus, in the numbers 2.405, 0.002405, and 240500, the digits 2, 4, 0, and 5 are significant figures. The zeros after the 5 in 240500 may or may not be significant figures.

When it is desired to indicate whether final zeros are significant or not, scientific notation is often used. Thus, in $2.405 \cdot 10^5$, the last significant figure is 5; in $2.40500 \cdot 10^5$, the final two zeros are regarded as significant.

To *round off* a number in which the last desired significant figure is in the units place or in any decimal place, drop all digits that lie to the right of the last significant figure. It is sometimes necessary also to increase the last digit in the retained part by 1.

If the first digit in the dropped part is less than 5, the last digit in the retained part is left unchanged. If the first digit in the dropped part is greater than 5 or if that digit is 5 and it is followed by digits other than 0, the last digit in the retained part is increased by 1. When the dropped part consists of the digit 5 alone or the digit 5 followed only by one or more zeros, we shall use the following procedure as an arbitrary rule in this book: If the last digit retained is odd, this digit is increased by 1; if it is even, it is left unchanged. This rule, although popular, is inferior

to common-sense rules in many cases. For example, if .245, .165, .485, and .725 are to be rounded off to two decimal places and then added, it would be more sensible to round off two of the numbers in one direction and two in the other direction.

To round off a number in which the last significant figure will lie to the left of the units place, first drop all digits to the right of the place occupied by the last significant figure, and replace each dropped digit to the left of the decimal point by a zero. Also, either leave the last digit of the retained part unchanged or increase that digit by one, in accordance with the directions just given for dropping only a decimal part. For example, if 2533.62 is to be rounded off to three significant figures, the result is 2530; and if 487,569 is to be rounded off to three significant figures, the result is 488,000.

There are two rules that are generally adopted by computers in working with approximate numbers in order to guard against retaining figures that may indicate a false degree of accuracy:

1. In adding or subtracting approximate numbers, round off the answer in the first place at the right in which any one of the given numbers ends.
2. In multiplying or dividing approximate numbers, round off the answer to the fewest significant figures found in any of the given numbers. The numbers entering a problem involving multiplication or division may be rounded off before the computation is begun. If these numbers are rounded off, they should have one more significant figure than the answer is to have.

While these rules point in the right direction, it should be mentioned that rounding off computed quantities to as many significant figures as there are in the given numbers does not necessarily produce the degree of accuracy implied by the results. The subject of accuracy of computation with approximate numbers is somewhat complicated and beyond the scope of a book at this level.

6–2. TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

One of the simplest, yet important, applications of trigonometry is in the solution of right triangles. A right triangle has, in addition to the 90° angle, five other parts. These are two acute angles and three sides. If we know the length of any two sides, or either acute angle and any one side, the triangle can be solved; that is, the unknown parts can be found.

To solve problems involving parts of triangles, we shall find it helpful to be able to express the trigonometric functions of an acute angle A of a right triangle ABC in terms of the sides of that right triangle. To derive suitable relationships, let us place the acute angle in standard position, as shown in Fig. 6–1.

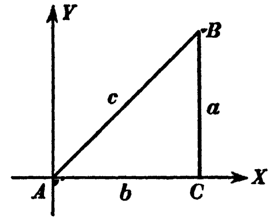

FIG. 6–1.

In the right triangle ABC the side AC, which is the abscissa of the point B, becomes the *side adjacent* to the angle A; the side CB, which is the ordinate of B, becomes the *side opposite* to angle A; and the side AB, which is the radius vector to B, becomes the *hypotenuse*. If we let the lengths of the side adjacent, the side opposite, and the hypotenuse be represented by the symbols b, a, and c, respectively, we may express the six functions of the acute angle A in terms of a, b, and c as follows:

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{c}, \tag{6–1}$$

$$\cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{b}{c}, \tag{6–2}$$

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{b}, \tag{6–3}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{c}{a}, \tag{6–4}$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{c}{b}, \tag{6–5}$$

$$\cot A = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{b}{a}. \tag{6–6}$$

By using (6–1) to (6–6), we can express the trigonometric functions of an angle of a right triangle without reference to any coordinate system, since the ratios of the sides remain the same regardless of the position of the triangle.

6–3. PROCEDURES FOR SOLVING RIGHT TRIANGLES

When solving a right triangle in which two parts are known, it is advisable to arrange the work systematically and to follow a definite procedure consisting of the following steps:

1. Draw a figure reasonably close to scale, and indicate the known parts.
2. Write an expression containing a trigonometric function which involves the two known parts and one unknown part.

3. Find the selected unknown part from this equation.
4. Find all other unknown parts of the triangle by a similar procedure.
5. Check all results.

Whenever possible, select a trigonometric function that gives a solution by means of a multiplication rather than a division.

In the following illustrative examples, the acute angles are represented by the letters A and B, and the right angle is denoted by C, while the small letters a, b, and c, respectively, represent the sides opposite them.

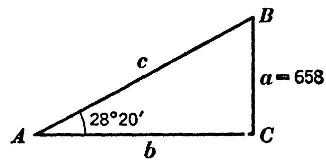

FIG. 6–2.

Example 6–1. Solve the triangle ABC, if $A = 28°20'$ and $a = 658$.

Solution: The triangle is drawn approximately to scale in Fig. 6–2. The unknown parts are the angle B and the sides b and c.

Since $A + B = 90°$, we have $B = 90° - 28°20' = 61°40'$.

To find the side b, we may apply either (6–3) or (6–6), since both equations involve the unknown b and the known parts A and a. We shall use $\cot A = \frac{b}{a}$ because it enables us to proceed to the solution by means of a multiplication rather than a division. Since $A = 28°20'$ and $a = 658$, we have

$$\frac{b}{658} = \cot 28°20'.$$

Then

$$\begin{aligned} b &= 658 \cot 28°20' \\ &= (658)\,(1.855) = 1220.59. \end{aligned}$$

In this example, we take b equal to 1221. This result is rounded off to four digits.

To find the side c, we shall use (6–4), or $\csc A = \frac{c}{a}\cdot$ We have, therefore,

$$\csc 28°20' = \frac{c}{658}\cdot$$

Hence,

$$\begin{aligned} c &= 658 \csc 28°20' \\ &= (658)\,(2.107) = 1386. \end{aligned}$$

To check, we may use the relation $\cos A = \dfrac{1221}{1386} = 0.8802$. Hence, $A = 28°20'$.

Checking by means of the Pythagorean theorem yields the result $b^2 = c^2 - a^2 = (c - a)\,(c + a) = (728)\,(2044) = 1488232$, whereas $b^2 = (1221)^2 = 1490841$. These values of b^2 agree when they are rounded off to three significant figures.

Note. In most situations where we must solve triangles, we are dealing with measured quantities, which are necessarily approximate. Therefore, our answers can be no more accurate than the

data we begin with. If the original data are approximate, our answers must be rounded off to the degree of accuracy indicated by the data.

In example 6-1, for instance, the answers may be given as $b = 1220$ and $c = 1390$, both rounded off to three significant figures.

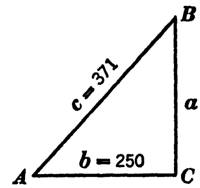

FIG. 6–3.

Example 6–2. Solve the triangle ABC, if $b = 250$ and $c = 371$.

Solution: The conditions are shown in Fig. 6–3. Since $\frac{b}{c} = \cos A$, we have $\cos A = \frac{250}{371} = 0.6738$. Therefore, $A = 47°38'$ and $B = 90° - 47°38' = 42°22'$.

To find a, we have a choice of combining the unknown a with either b or c; hence, we may use either $\frac{a}{b} = \tan A$ or $\frac{a}{c} = \sin A$. We shall illustrate, in order, the computation with each of these equations, thus providing a check on our work.

Using (6–3), we have

$$\frac{a}{250} = \tan 47°38'.$$

Hence, $a = 250 \tan 47°38' = (250)(1.096) = 274.0$, or 274 when rounded off to three figures.

Using (6–1), we have

$$\frac{a}{371} = \sin 47°38'.$$

Hence, $a = 371 \sin 47°38' = (371)(0.7388) = 274.09$ or 274 when rounded off to three figures.

EXERCISE 6–1

In each of the problems from 1 to 16, solve the right triangle.

1. $a = 12$, $A = 33°$.

2. $b = 168$, $A = 38°16'$.

3. $b = 62.4$, $B = 71°10'$.

4. $a = 42$, $c = 76$.

5. $a = 3.187$, $b = 6.249$.

6. $b = 63.21$, $B = 83°36'$.

7. $a = 4.318$, $B = 67°16'$.

8. $b = 827.6$, $c = 963.4$.

9. $a = 9.863$, $A = 36°21'$.

10. $b = 16.32$, $B = 87°10'$.

11. $b = 78.21$, $A = 43°17'$.

12. $a = 43.21$, $c = 63.75$.

13. $a = 123.6$, $b = 783.1$.

14. $a = 36.83$, $A = 57°44'$.

15. $b = 2.312$, $B = 40°57'$.

16. $a = 389.3$, $b = 62.34$.

17. A wire stretches from a point on level ground to the top of a vertical pole. It touches the ground at a point 15 feet from the foot of the pole and makes an angle of 63° with the horizontal. Find the height of the pole and the length of the wire.

18. A ladder 40 feet long rests against a vertical wall. If its foot is 5 feet from the base of the wall, what angle does it make with the ground?

19. A ladder 65 feet long is placed so that it will reach a window 35 feet above the ground on one side of a street. If the foot of the ladder is held in the same position and the top is moved to the other side of the street, it will reach a window 28 feet above the ground. How wide is the street from building to building?
20. The *grade* of a hill is the tangent of the angle the hill makes with the horizontal. Find the grade of a hill which is 275 feet long and which rises 120 feet.
21. To find the width of a river, a surveyor sights on a line across the river between two points A and B on opposite banks of the river. He then runs a line AC perpendicular to AB. He finds that AC is 250 feet and angle ACB is 42°17′. How wide is the river?
22. Find the length of a side of a regular hexagon and the radius of the inscribed circle, if the radius of the circumscribed circle is 10 feet.
23. An airplane rises 560 feet while flying upward for 2,387 feet along an inclined straight-line path. What is the angle of climb?
24. A pendulum 4.5 inches in length swings through an arc of 28°. How high does the bob rise above its lowest position?
25. A man 6 feet tall is walking along a straight horizontal path directly away from a lamp post 10.5 feet high. How far is he from the post at a certain instant when his shadow is 5 feet long?

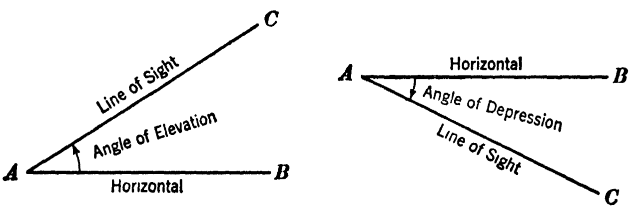

FIG. 6–4.

6–4. ANGLES OF ELEVATION AND DEPRESSION

Let the line AB in Fig. 6–4 be a level or horizontal line, and let an observer at the point A see an object at the point C. If the object C is above the horizontal line AB, then the angle BAC measured up from the horizontal to the line of sight AC is called the *angle of elevation* to C from A. If the object C is below the horizontal line AB, then the angle BAC measured down from the horizontal to the line of sight AC is called the *angle of depression* to C from A.

Example 6–3. From a point on the ground 300 feet from the base of a building, the angle of elevation to its top is 22°10′. How high is the building?

Solution: In Fig. 6–5, we have $b = 300$ and $A = 22°10'$. By (6–3), $\frac{a}{300} = \tan 22°10'$. Hence, $a = 300 \tan 22°10' = (300)\ (0.4074) = 122.22$. So the building is 122 feet high.

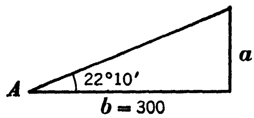

FIG. 6–5.

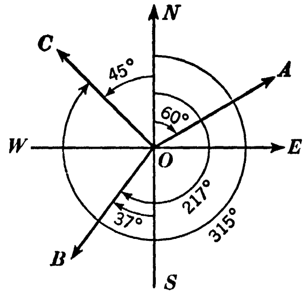

FIG. 6–6.

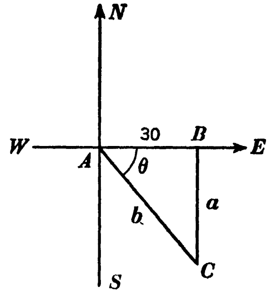

FIG. 6–7.

6–5. BEARING IN NAVIGATION AND SURVEYING

In marine and air navigation and in surveying, the direction in which an object is seen is expressed by the *bearing* or *azimuth* of the line of sight from the observer. The *bearing* of a line is the acute angle which its direction makes with a *meridian* or north-south line. Such angles are sometimes called *quadrant angles*, or *quadrant bearings*. To describe the bearing of a given direction, we first write the letter N or S, then the acute angle, and finally the letter E or W. The letters depend on the quadrant in which the given direction falls. Thus, the bearings of the lines OA, OB, and OC in Fig. 6–6 are N 60° E, S 37° W, and N 45° W, respectively.

The *azimuth* of a line differs from its bearing only in that the azimuth is the angle measured from 0° at north in a clockwise direction. An azimuth may have any value between 0° and 360°. Thus, in Fig. 6–6 the azimuths of the lines OA, OB, and OC are 60°, 217°, and 315°, respectively, measured clockwise from the north. This method of measuring directions is coming into more frequent use than that of *quadrant bearings*. We note also that the term *bearing* is often used instead of *azimuth*. Thus, we may speak of the *bearing of an object* regardless of whether we mean azimuth or bearing as here defined.

Example 6–4. A ship heads due east from a dock at a speed of 18 miles per hour. After traveling 30 miles it turns due south and continues at the same speed. Find its distance and bearing from the dock after 4 hours.

Solution: In Fig. 6–7, let A be the point at which the dock is located, let B be the point where the ship turns south, and let C be the position of the ship after 4 hours. Since the number of hours required to travel from A to B is $\frac{30}{18} = \frac{5}{3}$, the number of hours spent in travel from B to C is $4 - \frac{5}{3} = \frac{7}{3}$. Hence, $a = 18 \cdot \frac{7}{3} = 42$.

From the figure,

$$\tan \theta = \frac{42}{30} \quad \text{and} \quad \sec \theta = \frac{b}{30}.$$

Therefore,

$$\theta = 54°28',$$

and

$$b = 30 \sec 54°28' = (30)(1.721) = 51.6.$$

Hence, the distance from the dock is 52 miles and the bearing of the line AC is 144°28′ or S 35°32′ E.

6–6. PROJECTIONS

Often it is desirable to consider direction along a line segment. Thus, if P_1 and P_2 are the end points of a segment, we shall understand P_1P_2 to mean the *directed segment* from P_1 to P_2, the direction being specified by the order in which the end points are named. The non-negative length of the segment P_1P_2 is denoted by $|P_1P_2|$.

Frequently a directed segment P_1P_2 may lie on a line, such as a coordinate axis, on which a positive direction has been specified. Then the positive direction on the line may agree with the direction from P_1 to P_2, or the two directions may be opposite to each other. The *directed length* of the segment P_1P_2 is equal to $|P_1P_2|$ when the directions agree or when P_1 and P_2 coincide and is equal to $-|P_1P_2|$ when they disagree. Since the context will make the meaning clear, we shall designate the directed length of the segment P_1P_2 also by P_1P_2.

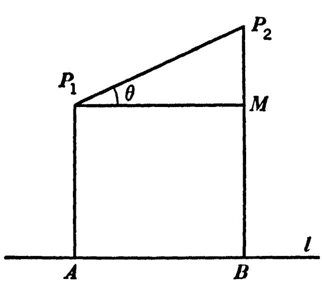

FIG. 6–8.

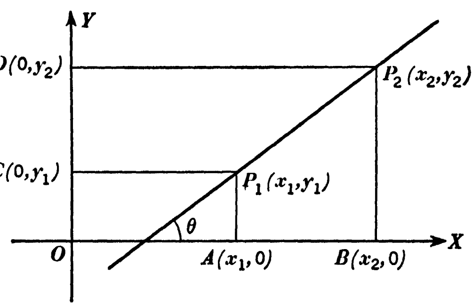

FIG. 6–9.

We recall that the *projection of a point on a given line* is the foot of the perpendicular dropped from the point to the line. If A in Fig. 6–8 is the projection of P_1 on the line l, and if B is the projection of P_2 on l, then the directed segment from A to B is the *projection on l of the directed segment* P_1P_2. We draw P_1M parallel to l, or perpendicular to P_2B, to show the angle θ between l and P_1P_2.

We shall now assume that a positive direction has been specified on the line l. Then, since $P_1M = AB$, it follows immediately from trigonometry that

$$AB = |P_1P_2| \cos \theta,$$

where θ is the acute angle between the positive end of l and the positive half-line determined by the directed segment P_1P_2. In Fig. 6–8 it is considered that l is positively directed toward the right.

The result just given can be applied, as seen in Fig. 6–9, in finding the projections of P_1P_2 upon the coordinate axes. The directed lengths of the projections upon the x-axis and the y-axis are, respectively,

$$AB = |P_1P_2| \cos \theta, \tag{6–7}$$

$$CD = |P_1P_2| \sin \theta, \tag{6–8}$$

where θ is the angle between OX and P_1P_2, as shown in Fig. 6–9.

When the coordinates of the end points of the segment P_1P_2 are known, the projections AB and CD are readily expressed in terms of these coordinates. From the definitions of horizontal and vertical distances given in Section 2–2, it follows that

$$AB = x_2 - x_1 \quad \text{and} \quad CD = y_2 - y_1. \tag{6–9}$$

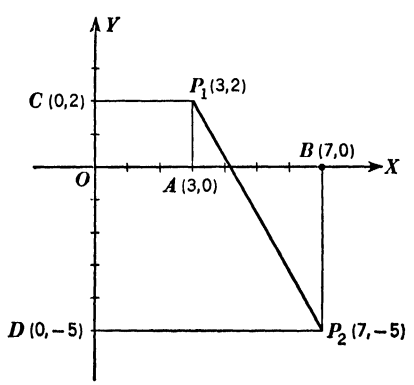

FIG. 6–10.

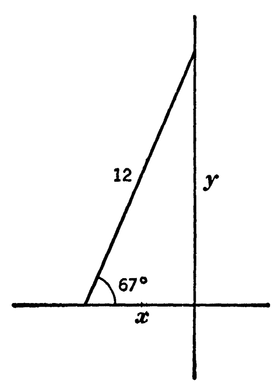

FIG. 6–11.

Example 6–5. What are the projections of the segment P_1P_2 on the axes, if $P_1 = (3, 2)$ and $P_2 = (7, -5)$?

Solution: In Fig. 6–10, $AB = x_2 - x_1 = 7 - 3 = 4$, and $CD = y_2 - y_1 = -5 - 2 = -7$. Since AB is $+4$, we know that AB is directed to the right. Also, since $CD = -7$, we know that CD is directed downward.

Example 6–6. A ladder 12 feet long leans against the side of a house and makes an angle of 67° with ground. Find its projections on the ground and on the side of the house.

Solution: The conditions are represented in Fig. 6–11. Let $l = 12$ be the length of the ladder. The required projections are found as follows.

The projection on the ground is given by

$$x = l \cos \theta = 12 \cos 67° = 12\ (0.3907) = 4.7.$$

The projection on the side of the house is given by

$$y = l \sin \theta = 12 \sin 67° = 12\ (0.9205) = 11.0.$$

EXERCISE 6–2

1. Two points A and B are 5,000 feet apart and at the same elevation. An airplane is 10,000 feet directly above point A. Find the angle of depression from a horizontal line through the airplane to point B and the airplane's distance from point B.
2. The Washington monument is approximately 555 feet high. Find the angle of elevation to the top of the monument from a point that is 621 feet from the base of the monument and at the same elevation as the base.
3. If a kite is 130 feet above the ground and 150 feet of string is out, find the angle of elevation to the kite, assuming the string to lie on a straight line.
4. Find the angle of elevation to the sun if a flagpole 95 feet high casts a shadow 63 feet long on horizontal ground.
5. A boat leaves its dock and heads N 52° W for 4 hours at 14 knots (1 knot = 1 nautical mile per hour = 6,080.4 feet per hour). It then turns and heads N 38° E for 3 hours at 16 knots. Find the boat's final bearing and distance from the dock.
6. The grade of a certain railroad bed is 0.1095. How many feet does a locomotive rise while traveling 175 feet along the track?
7. An approach must be built up to the end of a bridge which is 40 feet above ground. If the approach is to have a 10% grade and the original ground is assumed to be level, how far from the end of the bridge must the approach start?
8. A surveyor wishes to find the distance between two points A and B separated by a lake. He finds a point C on the shore of the lake such that angle ACB is 90°. He measures AC and BC and finds that AC is 640 feet and BC is 285 feet. How far apart are A and B?
9. A smokestack is 175 feet from a building. From a window of the building the angle of elevation to the top of the stack is 28°10′. The angle of depression to its base from the same window is 24°30′. Assuming that the ground is level, find *a*) the height of the window above the ground and *b*) the height of the smokestack.
10. Two ships leave the same port at the same time. One travels N 42° E at 25 knots. The other travels S 48° E at 33 knots. How far apart are the two ships after 4 hours?

6–7. SCALAR AND VECTOR QUANTITIES

We shall at this point find it necessary to distinguish carefully between two kinds of quantities, namely *scalar quantities* and *vector quantities*.

A *scalar quantity* is a quantity whose measure can be fully described by a number. It is a quantity which can be measured on a real number scale. For example, temperature is a scalar quantity, measured on the scale of a thermometer. Also we shall define the *scalar components* of the segment P_1P_2 to be the projections on the coordinate axes, or the directed lengths $x_2 - x_1$ and $y_2 - y_1$ given by (6–9) in Section 6–6. The student should note, however, that the scalar components of P_2P_1 are not equal to those of P_1P_2. The components of P_2P_1 are $x_1 - x_2$ and $y_1 - y_2$ and are the negatives of the respective components of P_1P_2.

A *vector quantity*, or simply a *vector*, is a quantity possessing both magnitude and direction. A vector may be represented by any one of a set of equal and parallel line segments. Algebraically, a vector is fully described by the scalar components of any segment representing it; all such segments have the same components. We shall, in fact, call these the *scalar components* of the vector and shall

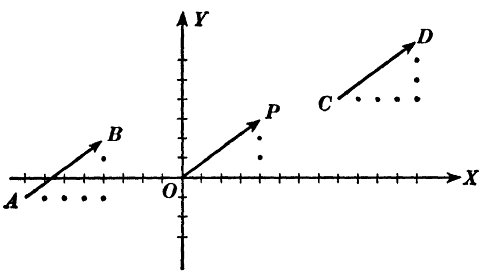

FIG. 6–12.

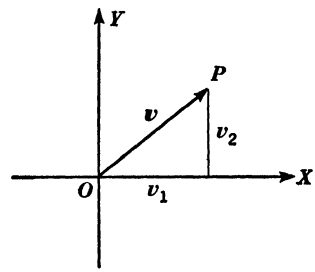

FIG. 6–13.

enclose them in brackets. In Fig. 6–12, three representations of the vector [4, 3] are shown. The arrowhead indicates the order in which the end points of the line segment are named.

We may denote a vector by a single letter in boldface type, say **v**, or may represent it geometrically by any one of the segments, such as AB, OP, or CD. Here A, O, and C represent the initial points of the three segments, while the terminal points at the arrowheads are B, P, and D. Actually, any other segment with the same magnitude and direction could have been selected to represent the vector **v**. Instead of using a single letter in boldface type to denote a vector represented by a segment P_1P_2, we may also use the notation $\overrightarrow{P_1P_2}$.

Properties of Vectors. We have seen that a vector is unambiguously represented by any one of a set of equal and parallel line segments. Hence, any point may be taken as the initial point of a segment which represents a vector. If the origin is so chosen, the coordinates of the end point $P(x, y)$ are actually the scalar components of the vector $\overrightarrow{OP}$. We can therefore give the following simple definition of the magnitude of a vector and its expression in terms of scalar components:

Definition. The *magnitude,* or *length,* of a vector $\mathbf{v}$ is the length of any one of the segments representing $\mathbf{v}$.

Let $\mathbf{v} = [v_1, v_2]$ be the vector represented by OP in Fig. 6–13, where v_1 and v_2 are scalar components. Then we have $|\mathbf{v}| = |\overrightarrow{OP}|$. Since OP is the hypotenuse of a right triangle,

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}. \tag{6–10}$$

In case the scalar components are given by the coordinates (x, y) of the end point P of the segment, the magnitude of the vector is $\sqrt{x^2 + y^2}$.

From the definition of a vector, it follows that two vectors are *equal* if and only if their respective scalar components are equal. For example, $\mathbf{u} = [u_1, u_2]$ equals $\mathbf{v} = [v_1, v_2]$ if and only if $u_1 = v_1$ and $u_2 = v_2$.

We also define a special vector $\mathbf{0} = [0, 0]$ to be the *zero vector.* It corresponds to the exceptional case in which P_2 coincides with P_1 and may be considered as represented geometrically by a segment of length zero, that is, by a point. The zero vector may be regarded as having any direction whatsoever.

If a vector $\mathbf{v} = [v_1, v_2]$ is given, the vector $-\mathbf{v} = [-v_1, -v_2]$ is defined to be the *negative* of $\mathbf{v}$. Thus, if $\overrightarrow{P_1P_2} = \mathbf{v}$ denotes a vector represented by the segment P_1P_2, then $\overrightarrow{P_2P_1} = -\mathbf{v}$ denotes a vector having the same length as P_1P_2 but oppositely directed, namely, from P_2 to P_1. We note that

$$-(-\mathbf{v}) = \mathbf{v}.$$

A *unit vector* is defined as a vector whose magnitude is unity. If $\mathbf{u} = [u_1, u_2]$ is any non-zero vector, then $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \left[\dfrac{u_1}{|\mathbf{u}|}, \dfrac{u_2}{|\mathbf{u}|}\right]$ is a unit vector.

Multiplication of a vector by a scalar is performed by multiplying the magnitude of the vector by the absolute value of the scalar, maintaining the direction of the vector if the scalar is non-negative and reversing it otherwise. Thus, by $k[u_1, u_2]$ we mean the vector

$[ku_1, ku_2]$. Using the equation $[u_1, u_2] = k\left[\frac{u_1}{k}, \frac{u_2}{k}\right]$, we can express any vector $\mathbf{v} = [u_1, u_2]$ as proportional to a unit vector $\mathbf{v}/k$ if we choose $k = \sqrt{u_1^2 + u_2^2}$. Changing $\mathbf{v}$ to a unit vector $\mathbf{v}/k$ is called *normalizing* $\mathbf{v}$.

Sums and Differences of Vectors. Of the many questions which arise in the study of vectors, the one of greatest importance for us at present concerns the addition of vectors. The *sum,* or *resultant,* of two vectors is defined to be the vector which has for its scalar components the sums of the scalar components of the two vectors. Thus, the sum of the vectors $\mathbf{u}$ and $\mathbf{v}$ is given by the relationship

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]. \tag{6–11}$$

It is important to note that some of the laws of the algebra of numbers also hold for vectors. Thus, the commutative law is

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

Also, the associative law of addition is

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

And, for every vector $\mathbf{v}$,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

That these laws are satisfied for vectors can be shown geometrically or can be seen from (6–11) by virtue of the known laws of addition of real numbers.

In terms of components, the rule for multiplication by a scalar is given by

$$k\mathbf{v} = [kv_1, kv_2].$$

In consequence of this definition, the following algebraic laws are satisfied:

$$\begin{aligned} k(\mathbf{u} + \mathbf{v}) &= k\mathbf{u} + k\mathbf{v}; \\ (k + m)\mathbf{u} &= k\mathbf{u} + m\mathbf{u}; \\ k(m\mathbf{u}) &= (km)\mathbf{u}. \end{aligned}$$

To find the sum of two vectors geometrically, we proceed as indicated in Fig. 6–14. This graphical representation of the sum of two vectors by means of the triangle construction was probably suggested by the behavior of physical quantities represented by vectors, such as forces, displacements, velocities, and accelerations. Their addition is effected by the *triangle law* or the *parallelogram law.*

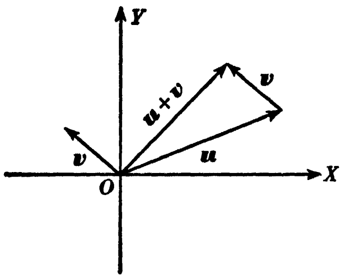

FIG. 6–14.

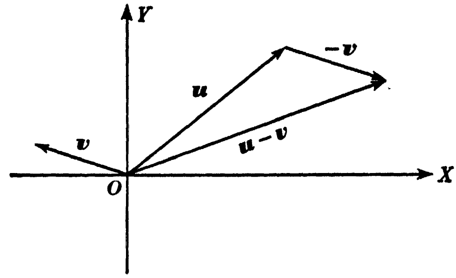

FIG. 6–15.

The following steps indicate the actual procedure.

1. Select a segment representing one of the vectors, say **u**, with its initial point at the origin of the coordinate system.
2. Place the initial point of a segment representing the second vector, say **v**, at the terminal point of the first segment.
3. Draw the segment from the origin to the terminal point of the segment for **v**. This segment represents the resultant vector, **u** + **v**.

The difference **u** − **v** of two vectors is defined in a manner analogous to that used in defining the difference of numbers. Thus,

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

If $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$, then

$$\mathbf{u} - \mathbf{v} = [u_1 - v_1, u_2 - v_2].$$

To find the difference **u** − **v** of two vectors geometrically, we proceed as indicated in Fig. 6–15. The steps are as follows:

1. Place segments representing **u** and **v** with their initial points at the origin.
2. Place the initial point of a segment representing −**v** at the terminal point of the segment representing **u**. Note that the segment representing −**v** is parallel and equal in length to that for **v**, but has the opposite direction.
3. Draw the segment from the origin to the terminal point of the segment for −**v**. This segment represents the vector **u** − **v**.

Example 6–7. Find the sum and difference of the vectors $\mathbf{u} = [5, -3]$ and $\mathbf{v} = [-2, 1]$.

Solution: The required vectors are

$$\mathbf{u} + \mathbf{v} = [5 - 2, -3 + 1] = [3, -2],$$

and

$$\mathbf{u} - \mathbf{v} = [5 - (-2), -3 - 1] = [7, -4].$$

Example 6–8. Express $\mathbf{u} = [4, 3]$ as proportional to a unit vector.

Solution: The expression representing the vector is

$$\mathbf{u} = [4, 3] = 5\left[\frac{4}{5}, \frac{3}{5}\right].$$

Here $5 = \sqrt{4^2 + 3^2}$ is a scalar multiplier of the unit vector $\left[\frac{4}{5}, \frac{3}{5}\right]$. To check, note that the magnitude of $\left[\frac{4}{5}, \frac{3}{5}\right]$ is

$$\left|\frac{\mathbf{u}}{|\mathbf{u}|}\right| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1.$$

To add vectors analytically, we first find the scalar components of the vectors to be added. From Section 6–7, we know that if segments representing $\mathbf{u}$ and $\mathbf{v}$ make angles α and β, respectively, with the x-axis, the scalar components are given by the projections upon the coordinate axes. Thus, as shown in Fig. 6–16,

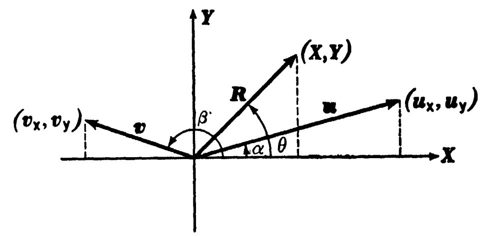

FIG. 6–16.

$$u_x = |\mathbf{u}| \cos\alpha, \quad u_y = |\mathbf{u}| \sin\alpha,$$

and

$$v_x = |\mathbf{v}| \cos\beta, \quad v_y = |\mathbf{v}| \sin\beta.$$

Then the components of the resultant will be given by the algebraic sums

$$X = u_x + v_x, \quad Y = u_y + v_y.$$

Thus, the magnitude of the resultant $\mathbf{R}$ is given by

$$|\mathbf{R}| = \sqrt{X^2 + Y^2}.$$

Also, the direction angle θ satisfies the relationship

$$\tan\theta = \frac{Y}{X}.$$

To determine the quadrant of θ correctly, it is important to keep in mind and use the correct signs of X and Y. For example, $\tan\theta = \frac{-1}{-1} = 1$ might lead one to an incorrect value 45° for θ instead of the correct third-quadrant angle 225°.

The student should note that this (analytic) procedure and the geometric procedure (parallelogram law) give the same results. This can be shown by appropriately combining the scalar components of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in Figs. 6–14 and 6–15, where we illustrated the geometric procedure.

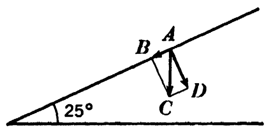

FIG. 6–17.

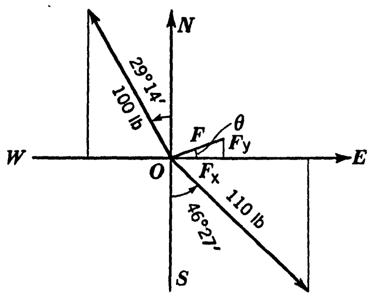

FIG. 6–18.

Example 6–9. A block weighing 500 pounds rests on a smooth plane making an angle of 25° with the horizontal, as indicated in Fig. 6–17. What force, parallel to the plane, is necessary to hold the block in position?

Solution: Let the block be at A. The force due to the weight may be represented by a vector acting vertically downward through A, such as vector $\overrightarrow{AC}$. The component of AC which is parallel to the inclined plane and which must be overcome is represented by the projection AB of AC on the inclined plane. Since angle $BAC = 90° - 25° = 65°$,

$$AB = 500 \cos 65° = (500)\ (0.4226) = 211.3.$$

Hence, a force of 211 pounds must be applied parallel to the inclined plane to keep the block from sliding. (It is assumed that three significant figures are appropriate.)

Example 6–10. Find the magnitude and direction of the resultant of a force of 110 pounds acting in the direction S 46°27′ E and a force of 100 pounds acting in the direction N 29°14′ W.

Solution: The given forces are represented by vectors in Fig. 6–18. Let F denote the magnitude of the resultant force and θ the angle it makes with OE.

Since $90° - 46°27' = 43°33'$ and $90° - 29°14' = 60°46'$, the components F_x and F_y of the resultant are found as follows:

$$\begin{aligned} F_x &= 110 \cos 43°33' - 100 \cos 60°46' \\ &= (110)\ (0.7248) - (100)\ (0.4884) \\ &= 79.73 - 48.84 = 30.89; \end{aligned}$$

$$\begin{aligned} F_y &= -\ 110 \sin 43°33' + 100 \sin 60°46' \\ &= -\ (110)\ (0.6890) + (100)\ (0.8726) \\ &= -\ 75.79 + 87.26 = 11.47. \end{aligned}$$

Now

$$\tan\theta = \frac{F_y}{F_x} = \frac{11.47}{30.89} = 0.3713, \quad \text{and} \quad \csc\theta = \frac{F}{11.47}.$$

Therefore $\theta = 20°22'$, and $F = 11.47 \csc 20°22' = (11.47)\ (2.873) = 32.95$. Hence, the magnitude of F is 33 pounds, and its direction is N 69°38′ E.

EXERCISE 6–3

1. Draw a diagram showing three different segments representing each of the given vectors. Find the magnitude of each vector.

a. [3, 4]. **b.** [12, 5]. **c.** [– 2, 4]. **d.** [– 3, – 5].
e. [1, – 1]. **f.** [5, 3]. **g.** [0, 4]. **h.** [0, – 1].

2. Express each of the vectors in Problem 1 in terms of a unit vector.
3. In each of the following cases, add the given vectors. Find the magnitude and direction of the resultant.

a. [1, 1] and [2, 3]. **b.** [4, – 2] and [1, – 10].
c. [2, 0] and [– 6, 3]. **d.** [1, 2], [4, 3], and [0, 7].

4. In Problem 3, parts (*a*), (*b*), and (*c*), subtract the first vector from the second, and find the magnitude and direction of the resultant.
5. In Problem 3(*d*), subtract twice the third vector from the sum of four times the first vector and twice the second vector, and find the magnitude and direction of the resultant.
6. A force of 40 pounds acts at an angle of 63° with the horizontal. What are the vertical and horizontal components of the force?
7. If a ship sails N 48° W at 30 knots, what are its westward and northward components?
8. One force of 28 pounds acts vertically upward on a particle. Another force of 43 pounds acts horizontally on the particle. What is the magnitude of the resultant force, and what is its direction?
9. A barrel weighing 160 pounds rests on a smooth plane which makes an angle of 22° with the horizontal. Find the force parallel to the plane necessary to keep the barrel from rolling down the plane.
10. Three forces act on a particle. One of 50 pounds makes an angle of 25° with the horizontal; a second of 60 pounds makes an angle of 50° with the horizontal; and the third of 75 pounds makes an angle of 230° with the horizontal. Find the magnitude and direction of the resultant force.
11. Four forces act on a body. The forces are 30, 45, 50, and 65 pounds, and they make angles with the horizontal of 25°, 160°, 240°, and 330°, respectively. Assuming that all the forces lie in the same vertical plane, find the magnitude and direction of the force necessary to hold the body in equilibrium. The required force is equal in magnitude and opposite to the resultant of the given forces.
12. A shell is fired at an angle of elevation of 37°. Its initial velocity is 2,500 feet per second. Find the horizontal and vertical components of its initial velocity.

6–8. LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

So far in this chapter, we have considered the use of a table of natural trigonometric functions and have solved various problems involving right triangles. In many problems, however, the computation is greatly facilitated by the use of logarithms to perform the numerical operations. For this purpose the values of the logarithms of the trigonometric functions are required. Table III might be used to obtain the logarithms of the functions found in Table II,

but the work is considerably lessened by the use of Table IV at the end of this book, which gives the logarithms of the trigonometric functions at once.

Table IV is a four-place table giving the logarithms of functions at intervals of 10 minutes from 0° to 90°. For the sine and cosine of any angle between 0° and 90°, the tangent of any angle between 0° and 45°, and the cotangent of any angle between 45° and 90°, the value of the function is less than 1; hence, the logarithms of these functions are negative, and we must write −10 after the tabulated entry. For the sake of uniformity, 10 has been added to each of the other entries in the table. In using the table, therefore, 10 must be subtracted from every entry.

The method of using Table IV is similar to that described for Table II and will be illustrated by the following examples.

Example 6–11. Find log sin 23°10′.

Solution: Since this angle is given in Table IV, we find that log sin 23°10′ = 9.5948–10.

Example 6–12. Find log cot 51°27′.

Solution: From Table IV we obtain the values for the following tabulation:

$$10' \left\{ \begin{array}{l} 7' \left\{ \begin{array}{l} \log\cot 51°20' = 9.9032 - 10 \\ \log\cot 51°27' = 9.9032 - 10 - x \end{array} \right\} x \\ \qquad \log\cot 51°30' = 9.9006 - 10 \end{array} \right\} 26$$

The tabular difference is 26. Since 51°27′ is $\frac{7}{10}$ of the way from 51°20′ to 51°30′, $x = 0.7(26) = 18.2$, and we have

$$\begin{aligned} \log\cot 51°27' &= (9.9032\text{–}10) - .0018 \\ &= 9.9014\text{–}10. \end{aligned}$$

Example 6–13. Find the acute angle θ if $\log\tan\theta = 9.7827\text{–}10$.

Solution: The positive part 9.7827 lies between the entries 9.7816 and 9.7845 in Table IV. The procedure for finding θ may be indicated as follows:

$$10' \left\{ \begin{array}{l} x \left\{ \begin{array}{l} \log\tan 31°10' = 9.7816 - 10 \\ \log\tan 31°(10+x)' = 9.7827 - 10 \end{array} \right\} 11 \\ \qquad \log\tan 31°20' = 9.7845 - 10 \end{array} \right\} 29$$

Hence, $\frac{x}{10} = \frac{11}{29}$, and

$$\begin{aligned} \theta &= 31°10' + \frac{11}{29}(10') \\ &= 31°14'. \end{aligned}$$

EXERCISE 6–4

In each of the problems from 1 to 15, find the value of the given logarithm.

1. log sin 48°20′. **2.** log sin 21°20′. **3.** log cos 86°20′.
4. log tan 88°30′. **5.** log cot 10°20′. **6.** log sec 43°50′.
7. log sin 13°26′. **8.** log sec 48°57′. **9.** log tan 41°14′.
10. log csc 78°32′. **11.** log cot 68°43′. **12.** log cos 18°18′.
13. log csc 83°16′. **14.** log cos 1°8′. **15.** log tan 51°34′.

In each of the problems from 16 to 35, find the angle (or angles) θ between 0° and 360°.

16. $\log \sin \theta = 8.8059–10$. **17.** $\log \tan \theta = 8.3669–10$.
18. $\log \cos \theta = 9.9959–10$. **19.** $\log \sin \theta = 9.1697–10$.
20. $\log \sec \theta = 0.4625$. **21.** $\log \cot \theta = 0.4882$.
22. $\log \tan \theta = 9.8483–10$. **23.** $\log \tan \theta = 0.1430$.
24. $\log \csc \theta = 0.4081$. **25.** $\log \sec \theta = 0.3586$.
26. $\log \sin \theta = 9.9567–10$. **27.** $\log \tan \theta = 9.7648–10$.
28. $\log \cos \theta = 9.9755–10$. **29.** $\log \cot \theta = 9.8666–10$.
30. $\log \sec \theta = 0.1967$. **31.** $\log \csc \theta = 0.3370$.
32. $\log \cos \theta = 9.1860–10$. **33.** $\log \cot \theta = 1.5976$.
34. $\log \sin \theta = 9.9974–10$. **35.** $\log \sec \theta = 0.3870$.

6–9. LOGARITHMIC SOLUTION OF RIGHT TRIANGLES

The solution of a right triangle by means of logarithms is exactly the same as by natural functions, except that for the actual numerical computation a table of logarithms of the natural functions is used in conjunction with a table of logarithms of numbers. The following example will illustrate the procedure.

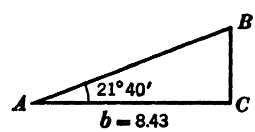

FIG. 6–19.

Example 6–14. Solve the right triangle ABC, in which $A = 21°40'$ and $b = 8.43$.

Solution: The values of the known parts are indicated in Fig. 6–19. We see that $B = 90° - 21°40' = 68°20'$. To find the side a, we have

$$\tan 21°40' = \frac{a}{8.43},$$

Hence,

$$a = 8.43 \tan 21°40',$$

or

$$\log a = \log 8.43 + \log \tan 21°40'.$$

Arrange the work as follows:

$$\begin{aligned} \log 8.43 &= 0.9258 && \text{(Table III)} \\ \log \tan 21°40' &= \underline{9.5991–10} && \text{(Table IV)} \\ \log a &= 10.5249–10 \end{aligned}$$

Therefore, $a = 3.35$. (Table III)

To find side c, we have

$$\frac{8.43}{c} = \cos 21°40'.$$

Hence,

$$c = \frac{8.43}{\cos 21°40'},$$

and

$$\log c = \log 8.43 - \log \cos 21°40'.$$

Arrange the work as follows:

$$\begin{aligned} \log 8.43 &= 10.9258\text{–}10 &&\text{(Table III)} \\ \log \cos 21°40' &= \underline{\ \ 9.9682\text{–}10} &&\text{(Table IV)} \\ \log c &= \ \ 0.9576 \end{aligned}$$

Therefore, $c = 9.07$. (Table III)

EXERCISE 6–5

In each of the problems from 1 to 8, solve the given right triangle.

1. $b = 100$, $A = 31°$. **2.** $c = 3.45$, $a = 1.76$. **3.** $A = 25°20'$, $a = 63.4$.
4. $A = 88°17'$, $c = 108.1$. **5.** $c = 6.275$, $B = 18°45'$. **6.** $a = 645.3$, $b = 396.3$.
7. $B = 27°9'$, $a = 36.13$. **8.** $b = 98.34$, $B = 18°48'$.

9. If a railroad track rises 30 feet in a horizontal distance of one mile, find the angle of inclination of the track.

10. A force of 341 pounds and another force of 427 pounds act at right angles to each other. Find the magnitude of the resultant force and the angle it makes with each of the forces.

11. A force of 628 pounds acts at 180°, and a force of 237 pounds acts at 270°. Find the direction and magnitude of the resultant.

12. An airplane is flying due east at a speed of 485 miles per hour, and the wind is blowing due south at 33.6 miles per hour. Find the direction and speed of the plane.

13. The westward and northward components of the velocity of an airplane are 363 and 487 miles, respectively. Find the direction and speed of the airplane.

14. The eastward and southward components of the velocity of a ship are 10.4 and 16.8 knots, respectively. Find the speed of the ship and the direction in which it is moving.

15. A force of 2673 pounds is acting at an angle of 47°13′ with the horizontal. Find its horizontal and vertical components.

16. A force of 162.4 pounds is just sufficient to keep a block at rest on a smooth inclined plane. If the block weighs 783.1 pounds, find the angle at which the plane is inclined to the horizontal.

17. Two tangents are drawn from a point P to a circle whose radius is 14.32 inches. If the angle between the tangents is 32°28′, how long is each tangent segment?

18. Two buildings of the same height are 11,640 feet apart. When an airplane is 8,000 feet above one of them, what is the angle of depression to the other one?

19. A cable which can withstand a pull of 10,000 pounds is used to pull loaded trucks up a ramp. If the angle of inclination of the ramp is 36°16′, find the weight of the heaviest truck which can be safely pulled up the ramp with the cable.

20. The angle of elevation from one point on level ground to the top of a flagpole is 45°28′. From a point on the ground 25 feet farther away the angle is 39°56′. How high is the pole?

7 Trigonometric Functions of Sums and Differences

7–1. DERIVATION OF THE ADDITION FORMULAS

Heretofore, we were concerned with relationships between trigonometric functions of a single angle. We shall now establish certain fundamental identities involving two angles, in terms of the functions of the single angles. The following identities express functions of the sum and difference of two angles in terms of the functions of the separate angles.

(7–1) $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

(7–2) $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

(7–3) $$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

(7–4) $$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

(7–5) $$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta},$$

(7–6) $$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

We shall now prove the formulas for the sine and cosine by using the derivation developed by E. J. McShane.[1]

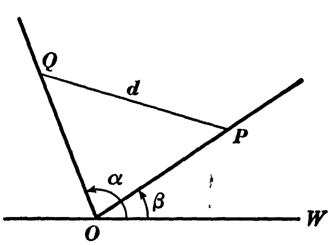

FIG. 7–1.

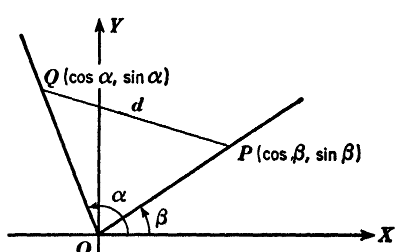

FIG. 7–2.

[1] E. J. McShane. "The Addition Formulas for the Sine and Cosine," American Mathematical Monthly, Vol. 48 (1941), pp. 688–89.

Let β and α be any two angles with the same initial side OW, as shown in Fig. 7–1. On their terminal sides we choose points P and Q, respectively, each at unit distance from O.

Let d represent the distance from P to Q. We shall now make two computations for d^2, using first OW, and then OP, as the x-axis.

When OW is used as the x-axis of a coordinate system, as shown in Fig. 7–2, we find that the coordinates of P and Q are $(\cos\beta, \sin\beta)$ and $(\cos\alpha, \sin\alpha)$, respectively. Hence, by the distance formula,

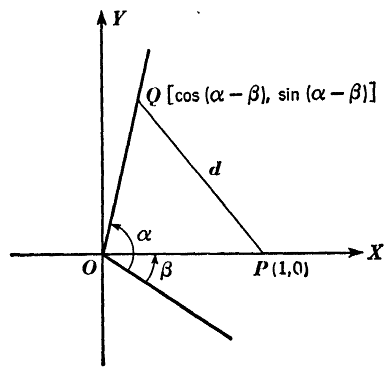

FIG. 7–3.

$$d^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$
$$= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta.$$

Since $\cos^2\alpha + \sin^2\alpha = \cos^2\beta + \sin^2\beta = 1$,

$$d^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

Let us now use OP as the x-axis, as shown in Fig. 7–3. Then the coordinates of P are $(1, 0)$ and those of Q are $[\cos(\alpha - \beta), \sin(\alpha - \beta)]$. Hence,

$$\begin{aligned} d^2 &= [\cos(\alpha - \beta) - 1]^2 + \sin^2(\alpha - \beta) \\ &= \cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta) \\ &= 2 - 2\cos(\alpha - \beta). \end{aligned}$$

Equating the two expressions for d^2 yields

$$2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha - \beta).$$

Therefore,

(7–4) $$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

This establishes (7–4).

Setting $\alpha = 90°$ in (7–4), we find that

(7–7) $$\cos(90^0 - \beta) = \sin\beta.$$

If in (7–7) we let $\beta = 90° - \gamma$, we have

(7–8) $$\sin(90^0 - \gamma) = \cos\gamma.$$

From (7–7), (7–8), and (7–4), we obtain

$$\begin{aligned} \sin(\alpha + \beta) &= \cos[90^0 - (\alpha + \beta)] \\ &= \cos[(90^0 - \alpha) - \beta] \\ &= \cos(90^0 - \alpha)\cos\beta + \sin(90^0 - \alpha)\sin\beta, \end{aligned}$$

or

(7–1) $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

This establishes (7–1).

Since $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$, we have, as a consequence of (7–4),

$$\begin{aligned}\cos(\alpha+\beta) &= \cos[\alpha-(-\beta)]\\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)\end{aligned}$$

or

(7–2) $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Similarly, from (7–1) it follows that

(7–3) $$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta.\end{aligned}$$

To prove (7–5), we use the relationship $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and (7–1) and (7–2). We then obtain

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing each term of the numerator and denominator by $\cos\alpha\cos\beta$, we have

$$\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

Hence,

(7–5) $$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

We may obtain (7–6) in a similar manner from (7–3) and (7–4), or from (7–5).

Example 7–1. Find the exact value of $\sin 75°$.

Solution: Substituting 45° for α and 30° for β in (7–1), we obtain

$$\begin{aligned}\sin 75° &= \sin(45°+30°)\\ &= \sin 45°\cos 30° + \cos 45°\sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}.\end{aligned}$$

Therefore,

$$\sin 75° = \frac{\sqrt{2}}{4}(\sqrt{3}+1).$$

Example 7–2. Find the exact value of tan $(\alpha + \beta)$ if α is a second-quadrant angle such that $\sin \alpha = \frac{3}{5}$, and β is a third-quadrant angle $\tan \beta = \frac{5}{12}$.

Solution: Let α be a second-quadrant angle for which $y = 3$ and $r = 5$. Hence, $x = -4$. It follows that $\tan \alpha = -\frac{3}{4}$. Using (7–5), we have

$$\tan(\alpha + \beta) = \frac{-\frac{3}{4} + \frac{5}{12}}{1 - \left(-\frac{3}{4}\right)\left(\frac{5}{12}\right)} = -\frac{16}{63}.$$

EXERCISE 7–1

In each of the problems from 1 to 8, find the exact value of the given function.

1. $\cos 75°$. **2.** $\tan 105°$. **3.** $\sin 135°$. **4.** $\sin 15°$.

5. $\tan 195°$. **6.** $\cos 195°$. **7.** $\tan 15°$. **8.** $\cos 105°$.

9. If α is a third-quadrant angle and β is a second-quadrant angle, and $\sin \alpha = -\frac{3}{5}$ and $\cos \beta = -\frac{5}{13}$, find $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$.

10. If $\tan \alpha = \frac{1}{2}$ and $\alpha - \beta = 45°$, find $\tan \beta$.

11. If $\tan \alpha = 3$ and $\alpha + \beta = 180°$, find $\tan \beta$.

12. If $\tan \alpha = \frac{3}{4}$ and $\tan \beta = \frac{1}{3}$, find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, where α and β are first-quadrant angles.

13. If $\cos \alpha = \frac{1}{2}$ and $\cos \beta = \frac{1}{3}$, find $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$, where α and β are acute angles.

14. If $\sin \alpha = \frac{4}{5}$, $\tan \beta = -\frac{5}{12}$, and α and β are both obtuse, find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

15. If $\sin \alpha = \frac{3}{5}$, $\cos \beta = \frac{24}{25}$, α is obtuse, and β is acute, find $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$.

16. If $\cos \alpha = -\frac{2}{3}$, $\sin \beta = \frac{1}{3}$, and α and β are obtuse, find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

17. If $\sin \alpha = \frac{3}{5}$, $\cos \beta = -\frac{5}{13}$, α is obtuse, and β is in the third quadrant, find $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$.

Prove each of the following identities:

18. $\sin(\alpha - 45°) = \frac{\sqrt{2}}{2}(\sin \alpha - \cos \alpha)$. **19.** $\cos(\alpha - \pi) = -\cos \alpha$.

20. $\frac{\sin(\alpha - \beta)}{\sin \alpha \sin \beta} = \cot \beta - \cot \alpha$. **21.** $\tan\left(\alpha + \frac{\pi}{4}\right) = \frac{1 + \tan \alpha}{1 - \tan \alpha}$.

22. $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2\cos^2 \alpha - 1 = 1 - 2\sin^2 \alpha$.

23. $\sin 2\alpha = 2 \sin \alpha \cos \alpha.$

24. $\dfrac{\sin 2\alpha}{\sin \alpha} - \dfrac{\cos 2\alpha}{\cos \alpha} = \sec \alpha.$

25. $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan \alpha.$

26. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta.$

27. $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}.$

28. $\dfrac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \dfrac{1+\tan\alpha\tan\beta}{1-\tan\alpha\tan\beta}.$

29. $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)} = \dfrac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}.$

30. $\cot(\alpha+\beta) = \dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}.$

31. $\dfrac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\cos^2\alpha\cos^2\beta} = \sec^2\alpha - \sec^2\beta.$

32. $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C$
$+ \cos A \cos B \sin C - \sin A \sin B \sin C.$

33. $\sin 3\alpha = \sin 5\alpha \cos 2\alpha - \cos 5\alpha \sin 2\alpha.$

34. $\cos(\alpha-\beta)\cos(\alpha+\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha.$

35. $\sin^2\left(\alpha+\dfrac{\pi}{4}\right) - \cos^2\left(\alpha+\dfrac{\pi}{4}\right) = \sin 2\alpha.$

7–2. THE DOUBLE-ANGLE FORMULAS

If we let $\beta = \alpha$ in (7–1), (7–2), and (7–5), we obtain functions of twice a given angle in terms of the functions of the angle itself. Thus, we have the following identities:

(7–9) $$\sin 2\alpha = 2\sin\alpha\cos\alpha,$$

(7–10) $$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha,$$

and

(7–11) $$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

We may obtain two other useful forms for $\cos 2\alpha$ from (7–10) by using in turn $\cos^2\alpha = 1 - \sin^2\alpha$ and $\sin^2\alpha = 1 - \cos^2\alpha$. These forms are given by the identities

(7–12) $$\cos 2\alpha = 1 - 2\sin^2\alpha,$$

and

(7–13) $$\cos 2\alpha = 2\cos^2\alpha - 1.$$

The following illustrations give an indication of the possible applications of (7–9), (7–10), and (7–11). The student should study them carefully.

$$\sin 4\alpha = \sin 2(2\alpha) = 2\sin 2\alpha\cos 2\alpha,$$

$$\sin\alpha = \sin 2\left(\frac{\alpha}{2}\right) = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2},$$

$$\cos\frac{\alpha}{3} = \cos 2\left(\frac{\alpha}{6}\right) = \cos^2\frac{\alpha}{6} - \sin^2\frac{\alpha}{6}.$$

Example 7–3. Find the exact value of sin 120° by means of a double-angle formula.

Solution: We use (7–9) to obtain

$$\sin 120^\circ = \sin 2(60^\circ) = 2 \sin 60^\circ \cos 60^\circ$$
$$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}.$$

Example 7–4. Derive a formula for $\cos 3\alpha$ in terms of $\cos \alpha$.

Solution: Applying the identity for $\cos(\alpha + \beta)$, and the double-angle formulas, we have

$$\begin{aligned}
\cos 3\alpha &= \cos(\alpha + 2\alpha) = \cos\alpha \cos 2\alpha - \sin\alpha \sin 2\alpha \\
&= \cos\alpha\,(2\cos^2\alpha - 1) - \sin\alpha\,(2\sin\alpha\cos\alpha) \\
&= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha \\
&= 2\cos\alpha\,(\cos^2\alpha - \sin^2\alpha) - \cos\alpha \\
&= 2\cos\alpha\,(2\cos^2\alpha - 1) - \cos\alpha \\
&= 4\cos^3\alpha - 3\cos\alpha.
\end{aligned}$$

Example 7–5. Prove the identity $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2.$

Solution: First combine the fractions on the left side and then reduce the result to the right side. Thus,

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta} = \frac{\sin(3\theta - \theta)}{\sin\theta\cos\theta}$$
$$= \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin 2\theta}{\sin 2\theta} = 2.$$

7–3. THE HALF-ANGLE FORMULAS

Functions of an angle in terms of the functions of twice that angle can be obtained directly from (7–12) and (7–13). If $\cos 2\alpha = 1 - 2\sin^2\alpha$ is solved for $\sin\alpha$, we obtain

$$\sin\alpha = \pm\sqrt{\frac{1 - \cos 2\alpha}{2}}.$$

Also, by solving $\cos 2\alpha = 2\cos^2\alpha - 1$ for $\cos\alpha$, we have

$$\cos\alpha = \pm\sqrt{\frac{1 + \cos 2\alpha}{2}}.$$

Since these formulas may be equally well regarded as expressing functions of half an angle in terms of the functions of the given angle itself, the same relationship is retained if the identities are written

(7–14) $$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}},$$

and

(7–15) $$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}.$$

These are the so-called *half-angle formulas* for the sine and cosine.

From (7–14) and (7–15) we obtain, by division,

$$\text{(7–16)} \qquad \tan\frac{\alpha}{2} = \frac{\sin\alpha/2}{\cos\alpha/2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

The algebraic signs in (7–14), (7–15), and (7–16) are determined by the quadrant of $\alpha/2$.

If we rationalize, in turn, the numerator and the denominator of the right hand side of (7–16), we obtain

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos^2\alpha}{(1+\cos\alpha)^2}} = \pm\sqrt{\frac{\sin^2\alpha}{(1+\cos\alpha)^2}},$$

or

$$\text{(7–17)} \qquad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}.$$

Similarly,

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{(1-\cos\alpha)^2}{1-\cos^2\alpha}} = \pm\sqrt{\frac{(1-\cos\alpha)^2}{\sin^2\alpha}}$$

or

$$\text{(7–18)} \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}.$$

The student will note that, in deriving (7–17) and (7–18), we have dropped the $\pm$ sign in each formula. The validity of this step should be verified by consideration of the signs of $\tan\alpha/2$, $\sin\alpha$, and $1 \pm \cos\alpha$. Thus, $\tan\alpha/2$ and $\sin\alpha$ necessarily have the same sign, while $1 \pm \cos\alpha$ is non-negative.

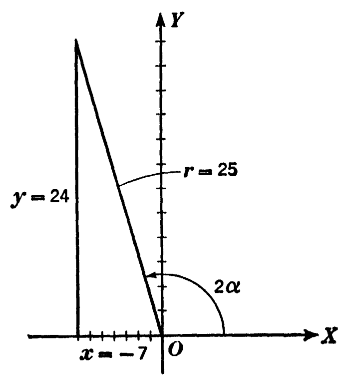

FIG. 7–4.

Example 7–6. Find the exact value of $\tan 22.5°$.

Solution: Since the exact values of the functions of 45° are known, we may use (7–16), (7–17) or (7–18). Selecting (7–18), we obtain

$$\tan 22.5° = \tan\frac{45°}{2} = \frac{1-\cos 45°}{\sin 45°} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$= \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1.$$

Example 7–7. Given $\tan 2\alpha = -\frac{24}{7}$, where 2α is a second-quadrant angle. Find $\sin \alpha$ and $\cos \alpha$.

Solution: Since 2α is an angle whose tangent is $-\frac{24}{7}$, we may find a point on the terminal side of the angle with $x = -7$ and $y = 24$, as shown in Fig. 7–4. Thus, $r = \sqrt{49 + 576} = 25$. Therefore,

$$\sin \alpha = \sqrt{\frac{1 + 7/25}{2}} = \frac{4}{5},$$

and

$$\cos \alpha = \sqrt{\frac{1 - 7/25}{2}} = \frac{3}{5}.$$

Example 7–8. Given that $\tan \frac{\alpha}{2} = u$, find $\sin \alpha$ and $\cos \alpha$ in terms of u.

Solution: Squaring both sides of (7–16), we have

$$\tan^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{1 + \cos \alpha}.$$

Substituting u for $\tan \frac{\alpha}{2}$, we get $u^2 = \frac{1 - \cos \alpha}{1 + \cos \alpha}$. If we solve this equation for $\cos \alpha$, we find that

$$\cos \alpha = \frac{1 - u^2}{1 + u^2}.$$

If we substitute this value of $\cos \alpha$ in the relationship $\sin^2 \alpha + \cos^2 \alpha = 1$, we obtain

$$\sin \alpha = \pm \sqrt{1 - \left(\frac{1 - u^2}{1 + u^2}\right)^2} = \frac{2u}{1 + u^2}.$$

Note that we can drop the $\pm$ sign, just as we did in (7–17) and (7–18), since $\tan \alpha/2$ and $\sin \alpha$ always have the same sign.

EXERCISE 7–2

In each of the problems from 1 to 8, find the exact functional value by using an appropriate double-angle or half-angle formula.

1. $\sin 22.5°$. **2.** $\cos 15°$. **3.** $\sin 120°$. **4.** $\cos 90°$.

5. $\sin 67.5°$. **6.** $\cos 67.5°$. **7.** $\tan 67.5°$. **8.** $\tan 60°$.

9. It is known that $\cos \theta = -\frac{12}{13}$ and θ is positive in the second quadrant. Find:

a. $\sin 2\theta$. **b.** $\cos \theta/2$. **c.** $\tan 2\theta$. **d.** $\cot 2\theta$.
e. $\sin \theta/2$. **f.** $\cos 2\theta$. **g.** $\sin 3\theta$. **h.** $\tan 4\theta$.

10. It is known that $\sin \theta = -\frac{40}{41}$ and θ is positive in the third quadrant. Find:

a. $\sin 2\theta$. **b.** $\cos \theta/2$. **c.** $\tan 2\theta$. **d.** $\cot 2\theta$.
e. $\sin \theta/2$. **f.** $\cos 2\theta$. **g.** $\sin 3\theta$. **h.** $\tan 4\theta$.

11. It is known that $\tan \theta = -3$ and θ is positive in the fourth quadrant. Find:

a. $\sin 2\theta$. **b.** $\cos \theta/2$. **c.** $\tan 2\theta$. **d.** $\cot 2\theta$.
e. $\sin \theta/2$. **f.** $\cos 2\theta$. **g.** $\sin 3\theta$. **h.** $\tan 4\theta$.

In each of the problems from 12 to 28, write the given expression in terms of a single function of a multiple of θ. Make use of appropriate formulas to reduce the answer to as few terms as possible.

12. $2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$. **13.** $\cos^2 \frac{3\theta}{2} - \sin^2 \frac{3\theta}{2}$. **14.** $2 \cos^2 3\theta - 1$.

15. $\frac{2 \tan 3\theta}{1 - \tan^2 3\theta}$. **16.** $1 - \sin^2 \frac{5\theta}{4}$. **17.** $\sin 2\theta \sqrt{\frac{1 - \cos 4\theta}{2}}$.

18. $\left(\sin \frac{\theta}{2} - \cos \frac{\theta}{2}\right)^2$. **19.** $\cos^4 \theta - \sin^4 \theta$. **20.** $\frac{\sin 2\theta}{\sin \theta} - \frac{\cos 2\theta}{\cos \theta}$.

21. $\frac{2}{\cot \theta - \tan \theta}$. **22.** $\frac{2 \cot \theta}{1 + \cot^2 \theta}$. **23.** $\frac{1 - \tan^2 \theta/2}{1 + \tan^2 \theta/2}$.

24. $\cot \frac{\theta}{2} + \tan \frac{\theta}{2}$. **25.** $\frac{\sin \theta/2}{1 - \cos \theta/2}$. **26.** $\frac{2 \tan \theta/2}{1 + \tan^2 \theta/2}$.

27. $\frac{1 - \cos 3\theta}{1 + \cos 3\theta}$. **28.** $\cos^2 (\theta^2) - \sin^2 (\theta^2)$.

Prove each of the following identities:

29. $\sin 4\theta = 4 \cos \theta(\sin \theta - 2 \sin^3 \theta)$. **30.** $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$.

31. $\sin 2\theta = (1 + \cos 2\theta) \tan \theta$. **32.** $1 + \sin 2\theta = (\sin \theta + \cos \theta)^2$.

33. $\frac{1 - \tan^2 \theta/2}{\cos \theta} = \frac{2(1 - \cos \theta)}{\sin^2 \theta}$. **34.** $4 \sin \theta \cos^2 \theta = \sin \theta + \sin 3\theta$.

7–4. PRODUCTS OF TWO FUNCTIONS EXPRESSED AS SUMS, AND SUMS EXPRESSED AS PRODUCTS

By adding and subtracting corresponding members of (7–1), (7–2), (7–3), and (7–4), we obtain

(7–19) $$\sin (\alpha + \beta) + \sin (\alpha - \beta) = 2 \sin \alpha \cos \beta,$$

(7–20) $$\sin (\alpha + \beta) - \sin (\alpha - \beta) = 2 \cos \alpha \sin \beta,$$

(7–21) $$\cos (\alpha + \beta) + \cos (\alpha - \beta) = 2 \cos \alpha \cos \beta,$$

(7–22) $$\cos (\alpha + \beta) - \cos (\alpha - \beta) = -2 \sin \alpha \sin \beta.$$

If we reverse these identities, they become the following *product formulas*, which express given products of sines and cosines as sums or differences:

(7–23) $$\sin \alpha \cos \beta = \frac{1}{2} [\sin (\alpha + \beta) + \sin (\alpha - \beta)],$$

(7–24) $$\cos \alpha \sin \beta = \frac{1}{2} [\sin (\alpha + \beta) - \sin (\alpha - \beta)],$$

(7–25) $$\cos \alpha \cos \beta = \frac{1}{2} [\cos (\alpha + \beta) + \cos (\alpha - \beta)],$$

(7–26) $$\sin \alpha \sin \beta = -\frac{1}{2} [\cos (\alpha + \beta) - \cos (\alpha - \beta)].$$

To obtain the *sum formulas,* which express given sums or differences of sines and cosines as products, we first let

$$\alpha + \beta = x \quad \text{and} \quad \alpha - \beta = y.$$

Then, solving for α and β, we have

$$\alpha = \frac{x+y}{2} \quad \text{and} \quad \beta = \frac{x-y}{2}.$$

Substituting these values of α and β in (7–19), (7–20), (7–21), and (7–22), we obtain the sum formulas. These are

(7–27) $$\sin x + \sin y = 2 \sin \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right),$$

(7–28) $$\sin x - \sin y = 2 \cos \left(\frac{x+y}{2}\right) \sin \left(\frac{x-y}{2}\right),$$

(7–29) $$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2}\right) \cos \left(\frac{x-y}{2}\right),$$

(7–30) $$\cos x - \cos y = -2 \sin \left(\frac{x+y}{2}\right) \sin \left(\frac{x-y}{2}\right).$$

Example 7–9. Express $\sin 3\alpha \cos 5\alpha$ as a sum of sines.

Solution: Using (7–24) and replacing α by 5α and β by 3α, we obtain

$$\cos 5\alpha \sin 3\alpha = \frac{1}{2}[\sin (5\alpha + 3\alpha) - \sin (5\alpha - 3\alpha)] = \frac{1}{2}[\sin 8\alpha - \sin 2\alpha].$$

Example 7–10. Express $\cos 4\theta + \cos 2\theta$ as a product of cosines.

Solution: Using (7–29) and replacing x by 4θ and y by 2θ, we have

$$\cos 4\theta + \cos 2\theta = 2 \cos \frac{4\theta + 2\theta}{2} \cos \frac{4\theta - 2\theta}{2} = 2 \cos 3\theta \cos \theta.$$

Example 7–11. Prove the identity $\dfrac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \tan x.$

Solution: $$\frac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \frac{2 \cos \left(\frac{7x+5x}{2}\right) \sin \left(\frac{7x-5x}{2}\right)}{2 \cos \left(\frac{7x+5x}{2}\right) \cos \left(\frac{7x-5x}{2}\right)}$$

$$= \frac{2 \cos 6x \sin x}{2 \cos 6x \cos x} = \tan x.$$

EXERCISE 7–3

In each of the problems from 1 to 10, write the given expression as a sum or difference of two sines or two cosines.

1. $\sin 3\theta \cos 4\theta$. **2.** $2 \sin 4\theta \cos 2\theta$. **3.** $2 \sin 6\theta \cos 4\theta$. **4.** $\sin \theta \cos 4\theta$.
5. $\cos 4\theta \cos 2\theta$. **6.** $2 \sin 65^\circ \cos 15^\circ$. **7.** $\sin 28^\circ \sin 20^\circ$. **8.** $\cos 21^\circ \cos 31^\circ$.
9. $\sin 5\theta \sin \theta$. **10.** $\sin 11\theta \sin 3\theta$.

In each of the problems from 11 to 20, write the given expression as a product of sines and cosines. Hint: In problems 18, 19, and 20, note that $\cos\theta = \sin(90° - \theta)$.

11. $\sin 3\theta + \sin 2\theta$.
12. $\cos\theta - \cos 4\theta$.
13. $\sin 6\theta + \sin 3\theta$.
14. $\sin 40° + \sin 20°$.
15. $\cos 80° - \cos 20°$.
16. $\sin 30° - \sin 80°$.
17. $\sin 40° + \sin 25°$.
18. $\sin 64° + \cos 38°$.
19. $\sin 40° + \cos 44°$.
20. $\sin 65° - \cos 33°$.

Prove each of the following identities:

21. $\sin\theta + \cos\theta = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$.

22. $\tan\alpha + \cot\beta = \frac{\cos(\alpha - \beta)}{\cos\alpha\sin\beta}$.

23. $\frac{\cos\theta}{\sec 4\theta} + \frac{\sin\theta}{\csc 4\theta} = \cos 3\theta$.

24. $\frac{\sin\theta}{\sec 3\theta} + \frac{\cos\theta}{\csc 3\theta} = \sin 4\theta$.

25. $\frac{\sin 2\theta + \sin 4\theta}{\cos 2\theta + \cos 4\theta} = \tan 3\theta$.

26. $\frac{\sin 2\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$.

27. $\frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\frac{\alpha + \beta}{2}$.

28. $\sin\left(\theta + \frac{\pi}{4}\right) + \sin\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}\sin\theta$.

29. $\sin\left(\theta + \frac{\pi}{3}\right) - \sin\left(\theta - \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$.

30. $\cos\left(\frac{\pi}{3} + \theta\right) - \cos\left(\frac{\pi}{3} - \theta\right) = -\sqrt{3}\sin\theta$.

31. $\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta + 1} = \tan\frac{\theta}{2}$.

32. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$.

33. $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$.

8 Graphs of Trigonometric Functions; Inverse Functions and Their Graphs

8–1. VARIATION OF THE TRIGONOMETRIC FUNCTIONS

In Section 3–2, the trigonometric functions were defined in terms of the coordinates (x, y) of the point $P(t)$, where the number t represents the directed length of the arc of a unit circle measured from the point $(1, 0)$. Later, in Section 3–9, an equivalent definition was given in terms of an angle θ in standard position. We shall now consider the variation of the trigonometric functions in the four quadrants as t, and with it θ, increases from 0 to 2π. From Fig. 8–1, we can read off the variations shown in Table 8–1.

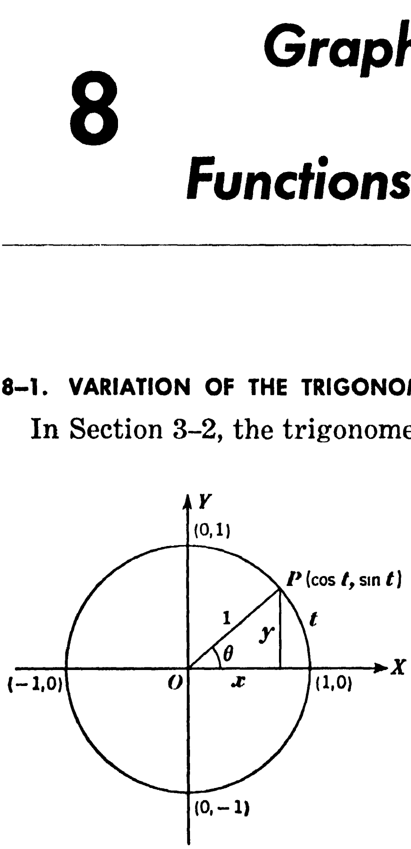

FIG. 8–1.

For example, by noticing the changes in y as t increases continuously from 0 to 2π, we find that sin t varies from 0 to 1 in the first quadrant, from 1 to 0 in the second, from 0 to -1 in the third, and from -1 to 0 in the fourth. Similar considerations lead to the results for the cosine and tangent.

Recalling that $\csc t = \dfrac{1}{\sin t}$, we know that if either of these functions increases the other decreases. Hence, the variation in csc t can be determined from the variation in sin t. Similarly, we may learn about the variation of sec t from that of cos t, and about the variation of cot t from that of tan t.

We found in Example 3–2 that tan $\pi/2$ is undefined, which means that tan t has no value when $t = \pi/2$. For the sake of easier tabu-

TABLE 8-1
Variation of Trigonometric Functions

	From	To	From	To	From	To	From	To
t	0	$\pi/2$	$\pi/2$	π	π	$3\pi/2$	$3\pi/2$	2π
$\sin t$	0	1	1	0	0	-1	-1	0
$\cos t$	1	0	0	-1	-1	0	0	1
$\tan t$	0	∞	$-\infty$	0	0	∞	$-\infty$	0
$\csc t$	∞	1	1	∞	$-\infty$	-1	-1	$-\infty$
$\sec t$	1	∞	$-\infty$	-1	-1	$-\infty$	∞	1
$\cot t$	∞	0	0	$-\infty$	∞	0	0	$-\infty$

lation of this result in Table 8–1 we have employed the much used symbols ∞ (infinity) and $-\infty$. These symbols merely signify that in the neighborhood of $\pi/2$ or one of its odd multiples the value of $\tan t$ is very large numerically. They are not to be used as numbers.

8–2. THE GRAPH OF THE SINE FUNCTION

To construct a graph representing the variation of the sine, we let x denote a real number or the value of an angle measured either in radians or in degrees, and we let y denote the corresponding value of the function. Corresponding values of x and y are plotted

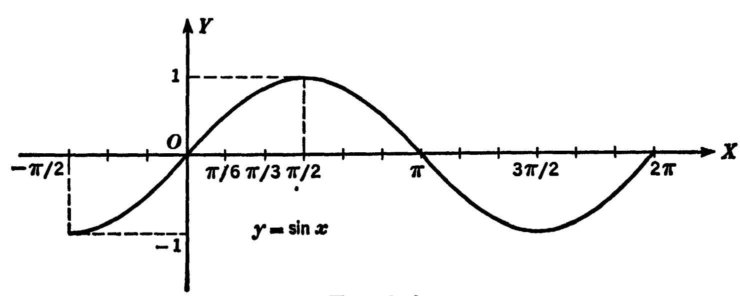

FIG. 8–2.

as points on a rectangular coordinate system. We can infer the general appearance of the curve from the results summarized in the preceding section, but an exact representation is more readily obtained by using a table of sines.

Let us now construct the graph of $y = \sin x$ from $x = -\pi/2$ to $x = 2\pi$. The following values, found by the methods of Section 3–2, are used to obtain the curve in Fig. 8–2.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
y	-1	-0.87	-0.5	0	0.5	0.87	1	0.87	0.5

x	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	-0.5	-0.87	-1	-0.87	-0.5	0

While the choice of a scale is arbitrary, a better proportioned graph results if the same unit of length is used on both axes. The unit so selected will represent the number 1 on the y-axis and one radian on the x-axis. In terms of this unit a suitable length can then be marked off on the x-axis to represent 2π or 360°.

8–3. THE GRAPHS OF THE COSINE AND TANGENT FUNCTIONS

Using the table of trigonometric functions, the student should make a table of corresponding values for $y = \cos x$ and one for

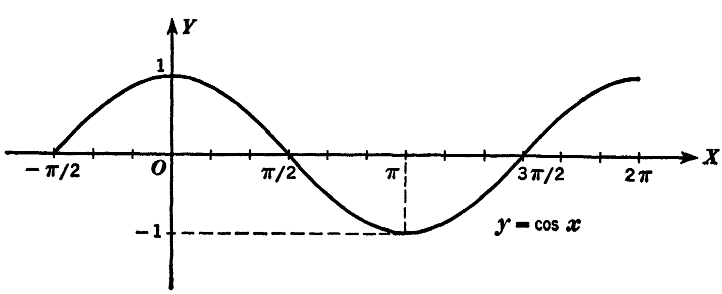

FIG. 8–3.

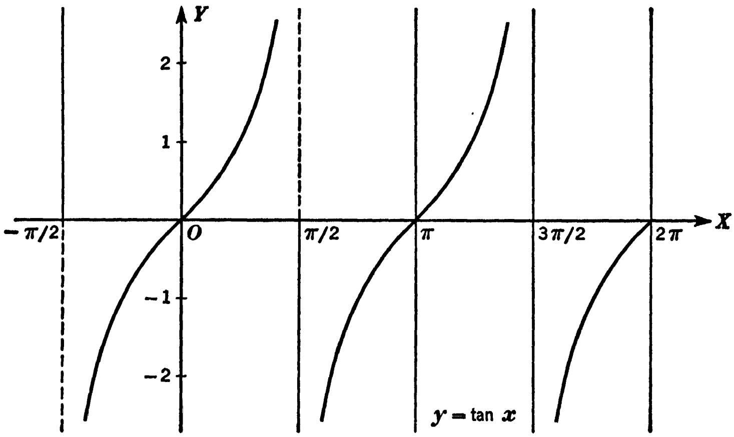

FIG. 8–4.

$y = \tan x$, similar to that used for $y = \sin x$ in Section 8–2. Study the graphs in Fig. 8–3 and Fig. 8–4 on the basis of the tables you have made.

If we compare Fig. 8–3 with Fig. 8–2, we see that the graph of $y = \cos x$ may be obtained from the graph of $y = \sin x$ by moving the graph of $y = \sin x$ to the left a distance of $\pi/2$ units. This fact can be checked by using the relationship $\cos x = \sin (x + \pi/2)$, from which it follows that $\cos 0 = \sin \pi/2$, $\cos \pi/6 = \sin 2\pi/3$, and so on.

8–4. PERIODICITY, AMPLITUDE, AND PHASE

The trigonometric functions are among the simplest of a large class of functions which are *periodic*. As a preliminary to defining periodic functions, we shall call attention to some examples of phenomena which recur periodically, such as the rotation of the earth about its axis, sound and water waves, the vibration of a spring, and many other vibratory and wavelike phenomena. The behavior of the object involved in a phenomenon of periodic nature determines the type of function that is required to represent it properly. We note particularly that, because of the recurrence characteristic of such a phenomenon, the values which the function assumes in any interval of given length are also taken on in any other interval of the same length. This statement apparently indicates that a function of x is *periodic with period* p if, for every value of x, the function returns to the same value when x is increased by p. More specifically, we state the following.

Definition. A function $f(x)$ is said to be *periodic* if there is a non-zero number p for which

$$f(x+p) = f(x)$$

for all numbers x in the domain of $f(x)$. Any such number p is called *a period;* the smallest positive number p satisfying the requirement is called *the period.*

Evidently, if p is *the* period of $f(x)$, then np is *a* period, for every integer n.

Periodicity of Trigonometric Functions. No matter which of the two viewpoints is considered in the definition of the trigonometric functions, we shall see that, if $0 \leqq t \leqq 2\pi$, then

$$\text{any trigonometric function of } (t + 2\pi) = \text{same function of } t.$$

According to the definitions given in Sections 3–1 and 3–2, $P(t)$ and $P(t + 2\pi)$ represent the same point on the circumference of the unit circle. To locate $P(t)$ we start at $(1, 0)$ and proceed around the circle in the proper direction a distance of $|t|$ units. To locate $P(t + 2\pi)$ we continue another 2π units from the point $P(t)$. This merely adds another complete revolution, and we arrive at the same point $P(t)$.

If we consider the definitions of the functions in terms of angles, as given in Section 3–9, we note that the angle $\theta + 2\pi$ is coterminal with θ and that any trigonometric function has the same value for coterminal angles.

Period of Sine, Cosine, Cosecant, and Secant. From a study of the sine curve, it is apparent that $\sin x$ assumes all values between -1 and $+1$ as x, starting from any value, varies through 2π units. In other words, the graph of the function repeats itself during each interval of length 2π, for positive and negative values of x. Or stated more concisely,

$$\sin(x + 2\pi) = \sin x.$$

This is equivalent to saying that $\sin x$ is a periodic function of x, and that 2π is *a* period. It remains now to show that 2π or $360°$ is the smallest positive number p for which $\sin(x + p) = \sin x$ and for which $\cos(x + p) = \cos x$.

Since, by definition of a period p, $\sin(x + p) = \sin x$ for any value x, we shall select for the purpose of our proof the particular value $x = \pi/2$. We then have

$$\sin\left(\frac{\pi}{2} + p\right) = \sin\frac{\pi}{2} = 1.$$

But since $\sin\left(\frac{\pi}{2} + p\right) = \cos p$, it follows that $\cos p = 1$.

Hence, p must be an even multiple of π. The smallest even multiple of π is 2π, which must also be the smallest positive period of the sine function.

In a similar manner, we find that 2π is also the period of the cosine function. Because of the reciprocal relationships existing between the sine and cosecant and between the cosine and secant, 2π is also the period of the cosecant and the secant.

Period of Tangent and Cotangent. To find the period of the tangent, we write

$$\tan (x + p) = \tan x,$$

and we let $x = 0$. Then $\tan p = 0$, and we find that π is the period of the tangent. A similar argument shows that π is also the period of the cotangent.

Period of sin bx. We have just seen that the period of $\sin x$ is 2π. We shall now determine the period of $\sin bx$, where b is a positive constant. That is, we want to know the smallest positive change in x which will produce a change of 2π in bx. If p represents this change in x, we can find p from the relationship

$$b(x + p) = bx + 2\pi.$$

Solving for p, we immediately find that

$$p = \frac{2\pi}{b}.$$

Thus, the period of $\sin bx$ is equal to the period of $\sin x$ divided by b, that is, $2\pi/b$.

Similarly, it can be shown that $2\pi/b$ is also the period of $\cos bx$, $\csc bx$, and $\sec bx$. It is also true that the period of $\tan bx$ and $\cot bx$ is π/b.

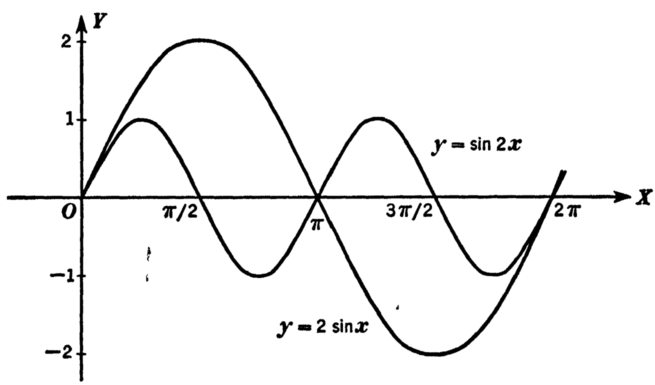

FIG. 8–5.

Let us consider the graph of $y = \sin 2x$ shown in Fig. 8–5. Since the period of $\sin 2x$ is $2\pi/b = 2\pi/2 = \pi$, the function will assume the same range of values in the interval from 0 to π that $\sin x$ takes in the interval from 0 to 2π.

Note that the graph of $y = 2 \sin x$ is similar in form to that of $y = \sin x$, which is shown in Fig. 8–2. The period is 2π for both curves. However, for any given value of x, the corresponding value of y in $y = 2 \sin x$ is twice as large as is the corresponding value of y in $y = \sin x$ for the same value of x. In the graph of $y = a \sin x$, where $a > 0$, the greatest value of y is a, and the smallest value of y is $-a$. The constant a is called the *amplitude* of the function or of the graph. Thus, the amplitude of the graph of $y = \sin x$ is 1, while that of the graph of $y = 2 \sin x$ is 2.

In general, for the function $y = a \sin x$, where a is a real number, the amplitude is equal to $|a|$ and the period is equal to 2π. Also, in general, for the graph of $y = a \sin bx$, where a and b are real, the amplitude is $|a|$ and the period is $2\pi/b$.

Phase Angle. Since the graph of $y = \cos x$ may be obtained from the graph of $y = \sin x$ by shifting it to the left a distance equal to $\pi/2$ units, we say that the graph of $y = \cos x$ *differs in phase* by $\pi/2$ from the graph of $y = \sin x$. The amount of horizontal displacement of two congruent graphs, amounting to $\pi/2$ radians in this case, is called the *phase difference,* or the *phase angle.* The amplitude and the period are the same for $y = \cos x$ as for $y = \sin x$.

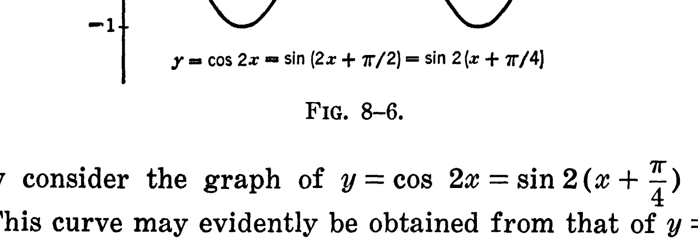

FIG. 8–6.

Now consider the graph of $y = \cos 2x = \sin 2(x + \frac{\pi}{4})$ in Fig. 8–6. This curve may evidently be obtained from that of $y = \sin 2x$ in Fig. 8–5 by a shift of $\pi/4$ units to the left in the x direction. Hence, the graph of $y = \sin (2x + \frac{\pi}{2})$ differs in phase from that of $y = \sin 2x$ by $\pi/4$ radians.

A simple method for finding the phase displacement is to locate the point near the origin for which the function $\sin\,(2x + \frac{\pi}{2})$ equals zero; that is, to find the smallest numerical value of x that makes the quantity $x + \frac{\pi}{4}$ zero. We have then $\sin 2(x + \frac{\pi}{4}) = 0$ when $x + \frac{\pi}{4} = 0$ or when $x = -\pi/4$. Hence, the phase difference is $\pi/4$ radians, and the shift of the graph is toward the left since the sign of x is negative.

It can be shown that, in general, the phase displacement of any trigonometric function of $(bx + c)$, where $b > 0$, is to the right or left by $|c/b|$ radians (or degrees). The direction of the displacement depends on whether c/b is negative or positive.

Finally, we arrive at the conclusion that for the graph of $y = a \sin\,(bx + c)$ the amplitude is $|a|$ and the period is $2\pi/b$. Also, its phase differs from that of $y = \sin x$ by c/b.

Because of its usefulness in many applications, we shall illustrate by means of an example a procedure for reducing an expression of the form $A \sin \theta + B \cos \theta$ to the form $a \sin\,(\theta + \alpha)$.

Example 8–1. Reduce $A \sin \theta + B \cos \theta$ to the form $a \sin\,(\theta + \alpha)$.

Solution: Write

$$A \sin \theta + B \cos \theta = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \sin \theta + \frac{B}{\sqrt{A^2 + B^2}} \cos \theta\right)\cdot$$

The absolute values of the coefficients $\frac{A}{\sqrt{A^2 + B^2}}$ and $\frac{B}{\sqrt{A^2 + B^2}}$ cannot be greater than 1, and the sum of their squares is 1. Hence, they may be taken as the cosine and sine, respectively, of some angle α. Therefore, the expression $A \sin \theta + B \cos \theta$ becomes

$$\sqrt{A^2 + B^2}\,(\cos \alpha \sin \theta + \sin \alpha \cos \theta) = \sqrt{A^2 + B^2} \sin\,(\theta + \alpha),$$

where $A = a \cos \alpha$ and $B = a \sin \alpha$.

Example 8–2. Express $4y = \sin 2\theta - \sqrt{3} \cos 2\theta$ in the form $y = a \sin\,(b\theta + \alpha)$, and sketch the graph.

Solution: Since $A = 1$ and $B = -\sqrt{3}$, we have $\sqrt{A^2 + B^2} = 2$. Therefore,

$$\sin 2\theta + \sqrt{3} \cos 2\theta = 2\left(\frac{1}{2} \sin 2\theta - \frac{\sqrt{3}}{2} \cos 2\theta\right).$$

If we identify this result with the expression $\sqrt{A^2 + B^2}\,(\cos \alpha \sin 2\theta + \sin \alpha \cos 2\theta)$, we have $\cos \alpha = \frac{1}{2}$ and $\sin \alpha = -\frac{\sqrt{3}}{2}\cdot$ It follows that α is a fourth quadrant angle and may be taken equal to $-\pi/3$.

We have, finally,

$$y = \frac{1}{4}(\sin 2\theta - \sqrt{3}\cos 2\theta) = \frac{1}{2}\sin\left(2\theta - \frac{\pi}{3}\right).$$

The amplitude of this function is $\frac{1}{2}$, its period is π, and its phase angle is $\pi/6$.

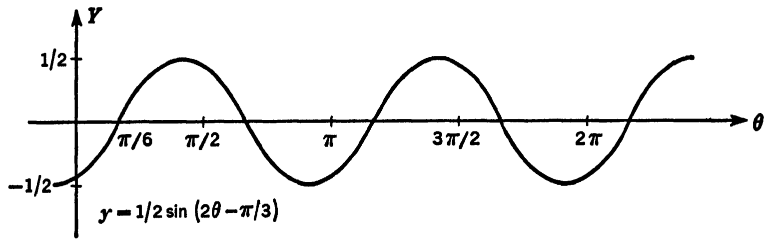

FIG. 8–7.

Since the interval from $(\pi/6, 0)$ to $(7\pi/6, 0)$ is one period in length, the part of the curve obtained for this interval may be repeated indefinitely in both directions to give the complete curve. The curve is shown in Fig. 8–7. (Here for convenience we have employed different units of length on the two axes.)

EXERCISE 8–1

In each of the problems from 1 to 24, find the period, amplitude, and phase angle of the trigonometric function.

1. $3 \sin \theta$. **2.** $\sin \frac{\theta}{3}$. **3.** $\frac{1}{2} \cos \theta$.

4. $\cos \frac{\theta}{2}$. **5.** $\frac{1}{3} \sin \frac{3}{4}\theta$. **6.** $2 \cot \frac{\theta}{2}$.

7. $4 \tan \frac{2}{5}\theta$. **8.** $\sin \frac{2\pi}{3}\theta$. **9.** $\cos \frac{\pi}{4}\theta$.

10. $\sin \pi\theta$. **11.** $3 \sin 5\theta$. **12.** $5 \tan \pi\theta$.

13. $\cot 6\theta$. **14.** $\sec 9\theta$. **15.** $\frac{1}{2} \cot 4\theta$.

16. $3 \csc \frac{3\pi}{4}\theta$. **17.** $\tan (\pi\theta + 4)$. **18.** $\cos (3\theta - 2)$.

19. $\csc (\pi\theta + 7)$. **20.** $\cot (2\pi\theta - \pi)$. **21.** $2 + \sin \theta$.

22. $6 - \cos 4\theta$. **23.** $5 + 3 \sin\left(2\theta - \frac{\pi}{6}\right)$. **24.** $4 + 2 \cos\left(2\theta + \frac{\pi}{3}\right)$.

In each of the problems from 25 to 30, sketch the graph of the given function by constructing a table of values.

25. $y = \cos \frac{1}{3}x$. **26.** $y = \tan \frac{1}{6}x$. **27.** $y = 2 \sin 3x$.

28. $y = 5 \cos \frac{1}{2}x$. **29.** $y = 2 \tan \frac{3}{4}x$. **30.** $y = 3 \sin 2x$.

Sketch each of the following graphs without constructing a table of values.

31. $y = \sin \frac{2}{3} x$. **32.** $y = \cos 4x$. **33.** $y = \tan \frac{1}{2} x$.

34. $y = 2 \cos 3x$. **35.** $y = 3 \cos \pi x$. **36.** $y = \frac{1}{2} \tan 3x$.

37. $y = \frac{1}{2} \sin \frac{1}{2} x$. **38.** $y = \frac{1}{3} \cos \frac{4}{3} x$. **39.** $y = 5 \cos \left(4x + \frac{\pi}{3}\right)$

40. $y = \cos (x + 2)$. **41.** $y = \sin \left(x - \frac{1}{2}\right)$. **42.** $y = \cos (3x - 2)$.

43. $y = \sin (2x + 1)$. **44.** $y = \cos (2\pi x - \pi)$. **45.** $y = \sin \left(\frac{3}{4} x + 7\right)$.

46. $y = \sin \theta + \cos \theta$. **47.** $y = \sin \theta - \cos \theta$.

48. $y = \sqrt{3} \sin 2\theta + \sqrt{5} \cos 2\theta$. **49.** $y = \frac{1}{2} \cos \pi\theta - \frac{3}{4} \sin \pi\theta$.

50. $y = \sqrt{2} \cos 3\theta - 3 \sin 3\theta$. **51.** $y = 2 \sin \theta - \cos 2\theta$.

8–5. INVERSE FUNCTIONS

In Section 2–3, we defined a function by setting up a rule of correspondence between two sets of numbers, X and Y, called the domain of definition of the function and the range set of the function, respectively. The function was called single-valued if just one number y of the set Y is assigned to each number x of the set X. If more than one number of Y corresponds to some value of x, the function is multiple-valued.

If we know that $y = f(x)$, we may pose a reverse problem. We assume y to be given and ask for all corresponding values of x. Naturally, y is limited to lie in the range of the given function, since otherwise no x exists. The function which makes correspond to each such y all values of x for which $y = f(x)$ is called the *inverse function* corresponding to the given function.

We shall begin our discussion of inverse functions with an example in which X is the set of all real numbers, Y is the set of all non-negative real numbers, and the correspondence is determined by the relationship

$$y = x^2.$$

Ordinarily, we assign values to x in order to calculate values of x^2. In this case, we have a rule of correspondence that assigns just one number y to each chosen number x. Hence, y is a single-valued function of x. The graph of $y = x^2$ is shown in Fig. 8–8.

Assume now that y is given and that we wish to determine corresponding values of x. To do this we solve the given equation $x^2 = y$ for x, and obtain two numbers $x = \sqrt{y}$ and $x = -\sqrt{y}$ corresponding to every non-negative number y. Since the rule of

correspondence assigns two values of x to each chosen number y, we see that x is a double-valued function of y. In this case, the admissible values of y are restricted to zero and the positive real numbers, while those of x comprise all real numbers, as was indicated in the specification for the sets X and Y.

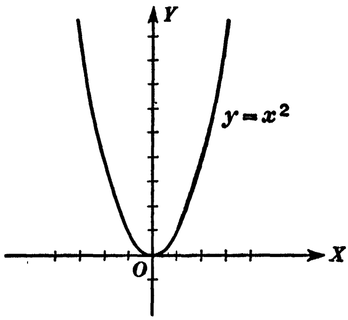

FIG. 8–8.

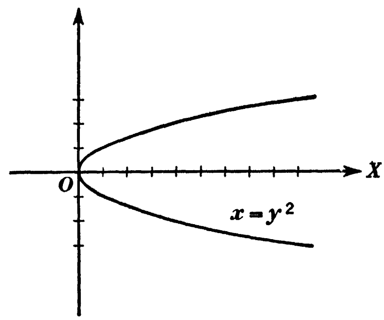

FIG. 8–9.

In the study of mathematics, we generally prefer to use the symbol x to represent the independent variable and y to represent the dependent variable. To be consistent with this preference, we shall call $y = x^2$ and $y = \pm\sqrt{x}$ inverse functions, each being called the inverse of the other.

The graph of $y = \pm\sqrt{x}$ is obtained by plotting that of $x = y^2$, as shown in Fig. 8–9. We note that the roles of x and y are interchanged in the two equations $y = x^2$ and $x = y^2$. Thus, we see that the curve in Fig. 8–9 is actually the curve of Fig. 8–8 with the axes interchanged and one of them reversed in direction.

8–6. INVERSES OF THE TRIGONOMETRIC FUNCTIONS

The Inverse Sine. Let us consider the function $y = \sin x$ and attempt to apply a discussion similar to that in Section 8–5. Here, as has been noted, the domain is the set of all real numbers, while the range is the interval $-1 \leqq y \leqq 1$. Referring to the graph of $y = \sin x$ in Fig. 8–2, and recalling the periodic properties of this graph, we see that, for every given number y such that $-1 \leqq y \leqq 1$, there are infinitely many values of x such that $y = \sin x$. To designate the totality of all values of x such that $y = \sin x$, we write

$$x = \sin^{-1} y,$$

which is read *x is the inverse sine of y.* The student should note carefully that the symbol $\sin^{-1} y$ must be distinguished from $(\sin y)^{-1}$, which equals $\dfrac{1}{\sin y}$ or $\csc y$.

Another notation that is frequently used to represent this inverse function is

$$x = \text{arc} \sin y,$$

which is read x *is the arc sine of* y.

In order to conform to the preferred practice of considering y as a function of x, we may designate the inverse of the sine function by writing

$$y = \sin^{-1} x \quad \text{or} \quad y = \text{arc} \sin x.$$

We note the following properties of this inverse function.

If x is a number such that $|x| > 1$, then y does not exist. This property follows from the fact that the sine function takes on only the values from -1 to 1. Hence, the inverse sine function $\sin^{-1} x$ is defined only when $-1 \leqq x \leqq 1$. For example, $\sin^{-1} 2$ is not defined, since there is no number or angle whose sine is 2.

If x is a number such that $|x| \leqq 1$, then y certainly exists. Moreover, because of the periodicity of the sine function, there are infinitely many values of $y = \sin^{-1} x$ corresponding to every such value of x. For example, if $x = 1/2$, then $y = \sin^{-1} 1/2$ means that y is any number or angle such that $\sin y = 1/2$. Then y may be taken as $\pi/6$, $5\pi/6$, or any value that differs from these by integral multiples of 2π. The totality of these values of $\sin^{-1} 1/2$ may be represented as

$$\frac{\pi}{6} + 2n\pi \text{ and } \frac{5\pi}{6} + 2n\pi, \text{ where } n = 0, \pm 1, \pm 2, \pm \cdots.$$

We shall find it convenient to plot the graph of $y = \sin^{-1} x$ for a further study of the inverse function. Since $y = \sin^{-1} x$ and $x = \sin y$ express exactly the same relation between x and y, the graph of Fig. 8–2 may be used as a graph of the inverse function. We obtain the graph of $y = \sin^{-1} x$ simply by interchanging the axes in Fig. 8–2 and reversing one of them in direction. The result is shown in Fig. 8–10.

The question now arises whether the y-axis can be subdivided into intervals within each of which y has just one value corresponding to each x such that $|x| \leqq 1$. One way of doing this is by selecting a first interval

$$-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}.$$

Other intervals, such as $\frac{\pi}{2} \leqq y \leqq \frac{3\pi}{2}$ and $\frac{-3\pi}{2} \leqq y \leqq -\frac{\pi}{2}$, are then selected. With the entire y-axis thus subdivided, we may think of the graph of $y = \sin^{-1} x$ as consisting of the graphs of infinitely many single-valued functions or *branches*.

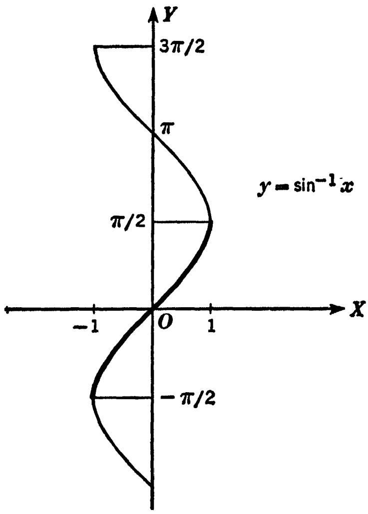

FIG. 8–10.

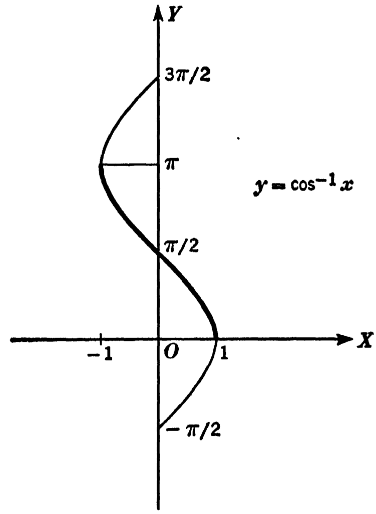

FIG. 8–11.

To avoid any ambiguity in later applications as a result of this multiple-valued property of the inverse sine, we shall often restrict y so as to make the function single-valued. There will then be but one value of y corresponding to each value of x such that $\sin y = x$. We shall determine this value from the branch for which $-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$. This branch is called the *principal branch.* The values of y chosen from this branch are called the *principal values* of the inverse-sine function and are represented by the equation

$$y = \text{Sin}^{-1} x \quad \text{or} \quad y = \text{Arc} \sin x.$$

Note that in this case the initial letter of the name is capitalized.

This restriction to the two quadrants containing the *smallest numerical values* of y results in a single-valued function $y = \text{Sin}^{-1} x$. The values of y are such that $-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$ for x in the interval $-1 \leqq x \leqq 1$. For example, we have $\text{Sin}^{-1}(-1) = -\pi/2$, $\text{Sin}^{-1}(-1/2) = -\pi/6$, $\text{Sin}^{-1} 0 = 0$, and $\text{Sin}^{-1} 1 = \pi/2$.

The Inverse Cosine. We shall next consider the function $y = \cos x$. With the help of Fig. 8–3, we see that the range of $y = \cos x$ is the interval $-1 \leqq y \leqq 1$, and that for every y in this interval there are infinitely many values of x such that $y = \cos x$. We are thus led to the inverse cosine function, which makes correspond to y all the values of x such that $y = \cos x$. If again we interchange the symbols x and y, we may write the inverse cosine function as

$$y = \cos^{-1} x \quad \text{or} \quad y = \text{arc} \cos x.$$

This inverse function has the following properties.

If x is a number such that $|x| > 1$, then y does not exist, because there is no value of y for which $|\cos y| > 1$.

If x is a number such that $|x| \leqq 1$, then there are infinitely many values of y designated by $\cos^{-1} x$.

The graph of $y = \cos^{-1} x$ is shown in Fig. 8–11. The method for plotting it is similar to that used for graphing $y = \sin^{-1} x$. We first write $x = \cos y$. We then plot a cosine curve by proceeding as for Fig. 8–3, except that values of the independent variable y are laid off on the y-axis.

The *principal branch* of the curve in Fig. 8–11 is the portion of the curve for which $0 \leqq y \leqq \pi$. It is represented by the *principal value* of the function, which is denoted by

$$y = \text{Cos}^{-1} x \quad \text{or} \quad y = \text{Arc} \cos x.$$

The Inverse Tangent. The inverse tangent function is

$$y = \tan^{-1} x \quad \text{or} \quad y = \text{arc} \tan x.$$

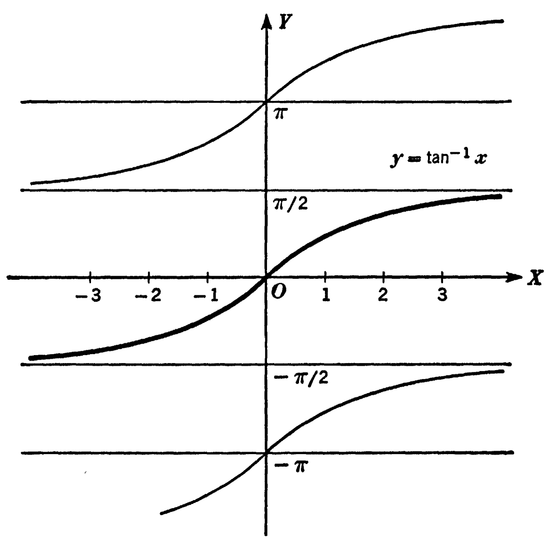

FIG. 8–12.

It is represented by the graph in Fig. 8–12. We note that there are infinitely many values of y for every value of x.

The principal branch of the graph of $y = \tan^{-1} x$ is the portion for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$ · This is represented by the equation

$$y = \text{Tan}^{-1} x \quad \text{or} \quad y = \text{Arc} \tan x.$$

The Inverse Cotangent, Secant, and Cosecant. The other inverse trigonometric functions are:

$$y = \cot^{-1} x \quad \text{or} \quad y = \text{arc} \cot x,$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \text{arc} \sec x,$$
$$y = \csc^{-1} x \quad \text{or} \quad y = \text{arc} \csc x.$$

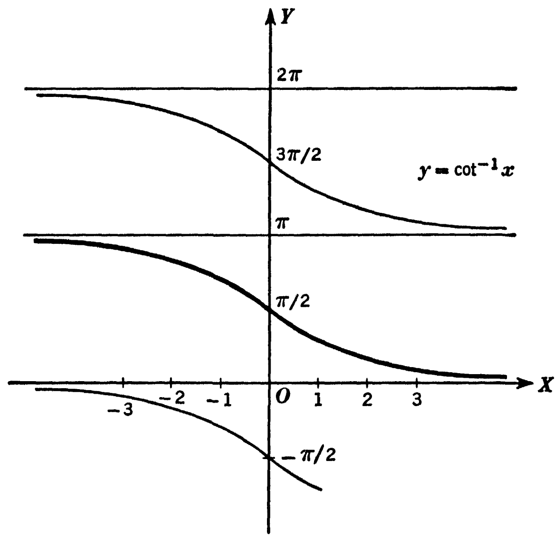

FIG. 8–13.

We shall show only the graph of $y = \cot^{-1} x$. See Fig. 8–13. The principal branch of this curve is given by $0 < y < \pi$.

Principal Values of the Inverse Cosecant and Secant. The selection of principal values of $y = \csc^{-1} x$ and $y = \sec^{-1} x$ is by no means uniform among all authors. Some writers adopt the range between 0 and $\pi/2$ for both functions when x is positive, and between $-\pi$ and $-\pi/2$ when x is negative. The authors prefer, however, to use the definitions

$$\text{Csc}^{-1} x = \text{Sin}^{-1} \left(\frac{1}{x}\right),$$

and

$$\text{Sec}^{-1} x = \text{Cos}^{-1} \left(\frac{1}{x}\right).$$

We have, therefore, the following ranges of principal values of the inverse trigonometric functions:

$$-\frac{\pi}{2} \leqq \text{Sin}^{-1} x \leqq \frac{\pi}{2}, \qquad 0 \leqq \text{Cos}^{-1} x \leqq \pi,$$

$$-\frac{\pi}{2} < \text{Tan}^{-1} x < \frac{\pi}{2}, \qquad 0 < \text{Cot}^{-1} x < \pi,$$

$$-\frac{\pi}{2} \leqq \text{Csc}^{-1} x \leqq \frac{\pi}{2}\ (\text{Csc}^{-1} x \neq 0), \qquad 0 \leqq \text{Sec}^{-1} x \leqq \pi \left(\text{Sec}^{-1} x \neq \frac{\pi}{2}\right)$$

Example 8–3. Find the value of $\cos\left(\text{Sin}^{-1}\frac{3}{5}\right)\cdot$

Solution: Let $\text{Sin}^{-1}\frac{3}{5} = \theta$. Then $\sin\theta = \frac{3}{5}$, and $\cos\theta = \pm\frac{4}{5}\cdot$ Since only the principal value of $\sin^{-1}\frac{3}{5}$ is used, θ is a first-quadrant angle and $\cos\theta$ cannot equal $-\frac{4}{5}\cdot$ Hence, $\cos\left(\text{Sin}^{-1}\frac{3}{5}\right) = \frac{4}{5}\cdot$

Example 8–4. Find the value of $\sin[\text{Tan}^{-1}(-x)]$, x being a positive number.

Solution: Let $\text{Tan}^{-1}(-x) = \theta$. Then $\tan\theta = -x$, and θ lies between $-\pi/2$ and 0. If angle θ is constructed in standard position, as shown in Fig. 8–14, then $\sin\theta$ is found to be $\frac{-x}{\sqrt{1+x^2}}\cdot$ Hence, $\sin[\text{Tan}^{-1}(-x)] = \frac{-x}{\sqrt{1+x^2}}\cdot$

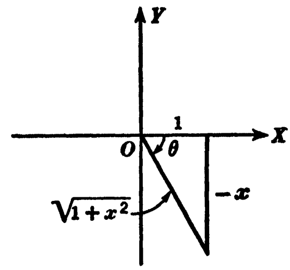

FIG. 8–14.

Example 8–5. Find Arc cos (cot 60°).

Solution: Let Arc cos (cot 60°) $= \theta$. Then $\cos\theta = \cot 60° = .5774$, by Table II. Therefore, the value of θ is Arc cos .5774. Since θ is restricted to the interval from 0° to 180°, θ must, in this case, be a first-quadrant angle. Hence, $\theta = \text{Arc cos } .5774 = 54°44'$, and Arc cos (cot 60°) $= 54°44'$.

EXERCISE 8–2

In each of the problems from 1 to 6, find the inverse of the given function.

1. $y = \frac{2x-5}{3}\cdot$ **2.** $y = \frac{2x-1}{6}\cdot$ **3.** $y = mx + b$.

4. $y = \frac{4}{15}x + \frac{7}{6}\cdot$ **5.** $y = \frac{2x-1}{3} - \frac{3x+6}{4}\cdot$ **6.** $y = 2x - \frac{3}{2}\cdot$

In each of the problems from 7 to 22, find the value of the given expression.

7. $\text{Sin}^{-1}\frac{1}{2}\cdot$ **8.** $\text{Tan}^{-1} 1$. **9.** Arc cos $\frac{\sqrt{3}}{2}\cdot$ **10.** $\text{Cos}^{-1}(-1)$.

11. $\text{Cot}^{-1} 0$. **12.** $\text{Sin}^{-1}\left(-\frac{1}{2}\right)\cdot$ **13.** $\text{Cot}^{-1}\frac{\sqrt{3}}{3}\cdot$ **14.** $\text{Csc}^{-1} 1$.

15. $\text{Tan}^{-1}\frac{\sqrt{3}}{3}\cdot$ **16.** $\text{Cos}^{-1}\left(\frac{-\sqrt{2}}{2}\right)\cdot$ **17.** $\text{Tan}^{-1}(-\sqrt{3})$. **18.** Arc tan (-1).

19. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)\cdot$ **20.** $\text{Tan}^{-1}[\sin(-\pi/2)]$.

21. $\text{Sin}^{-1}(-0.414)$. **22.** $\text{Tan}^{-1}(-1.414)$.

In each of the problems from 23 to 37, evaluate the given expression.

23. $\tan\left(\text{Sin}^{-1}\frac{12}{13}\right)\cdot$ **24.** $\sin\left(\text{Sin}^{-1}\frac{1}{2}\right)\cdot$ **25.** $\sin\left(\text{Tan}^{-1}\frac{5}{12}\right)$

26. $\cot\left(\frac{1}{2}\text{Sin}^{-1}\frac{\sqrt{3}}{2}\right)$ **27.** $\cos\left(\text{Cos}^{-1}\frac{3}{4}\right)\cdot$ **28.** $\tan(\text{Sin}^{-1} .6450)$.

29. $\sin(\text{Cos}^{-1} .9200)$. **30.** $\tan(\text{Cot}^{-1} 2)$. **31.** $\sec\left(\text{Sin}^{-1}\frac{1}{5}\right)$.

32. $\sin\left(\text{Cot}^{-1}\frac{1}{4}\right)$. **33.** $\cot\left(\text{Cos}^{-1}\frac{3}{5}\right)$. **34.** $\sin\left(\text{Cot}^{-1}\frac{2}{3}\right)$.

35. $\cos\left[\text{Cot}^{-1}\left(-\frac{3}{4}\right)\right]$. **36.** $\cos\left(\text{Sin}^{-1}\frac{2}{3}\right)$. **37.** $\text{Sin}^{-1}\left(\tan\frac{3\pi}{4}\right)$.

In each of the problems from 38 to 56 simplify the given expression, taking u as a positive number. In 38 to 40, 54, 56, $u < \pi/2$. In 44, 46, 48, 55, $u < 1$.

38. $\text{Cos}^{-1}(\sin u)$. **39.** $\text{Sin}^{-1}(-\sin u)$.

40. $\text{Sin}^{-1}(\cos u)$. **41.** $\csc\left(\text{Sin}^{-1}\frac{1}{u}\right)$.

42. $\sin(\text{Sin}^{-1} u)$. **43.** $\tan(\text{Tan}^{-1} u)$.

44. $\sec(\text{Sin}^{-1}\sqrt{1-u^2})$. **45.** $\cot\left(\text{Tan}^{-1}\frac{u}{\sqrt{1+u^2}}\right)$.

46. $\tan\left(\text{Cot}^{-1}\frac{u}{\sqrt{1-u^2}}\right)$. **47.** $\tan\left(\text{Sin}^{-1}\frac{u}{\sqrt{u^2+1}}\right)$.

48. $\csc(\text{Sin}^{-1}\sqrt{1-u^2})$. **49.** $\tan\left(\text{Cos}^{-1}\frac{1}{\sqrt{1+u^2}}\right)$.

50. $\cos(\text{Tan}^{-1} u)$. **51.** $\sec(\text{Cos}^{-1} u)$.

52. $\cot(\text{Sin}^{-1} u)$. **53.** $\sec(\text{Sin}^{-1} u)$.

54. $\text{Sin}^{-1}(\sin u)$. **55.** $\text{Cos}^{-1}(\cos\sqrt{1-u^2})$.

56. $\text{Cot}^{-1}\left(\tan\frac{u}{\sqrt{1+u^2}}\right)$.

In each of the problems from 57 to 65, draw the graph of the given function by changing from the inverse function to the direct function and using the period, amplitude, and phase angle of the function to assist in plotting.

57. $y = \frac{1}{2}\sin^{-1} x$. **58.** $y = 2\tan^{-1} x$.

59. $y = \frac{3}{4}\cos^{-1} x$. **60.** $y = \frac{1}{2}\sin^{-1} x - 1$.

61. $y = \frac{1}{3}\cos^{-1} x + 2$. **62.** $y = 4\tan^{-1} x + 3$.

63. $y = \frac{1}{\pi}(\text{Cos}^{-1} 2x + 1)$. **64.** $y = 3\sin^{-1}(2x+1) - 2$.

65. $y = 4\tan^{-1}\left(\frac{x+1}{2}\right)$.

66. Prove that $\sin(\text{Cos}^{-1} u) \geqq 0$, if $0 < u \leqq 1$.
67. Prove that $\cos(\text{Sin}^{-1} u) \geqq 0$, if $0 < u \leqq 1$.
68. Prove that $\cos(\text{Tan}^{-1} u) > 0$, if $u \geqq 0$.
69. Prove that $\tan(\text{Cos}^{-1} u) \geqq 0$, if $0 < u \leqq 1$.
70. Prove that $\text{Sin}^{-1} u + \text{Sin}^{-1}(-u) = 0$, if $-1 \leqq u \leqq 1$.
71. Prove that $\text{Tan}^{-1} u + \text{Tan}^{-1}(-u) = 0$ for all values of u.
72. Prove that $\text{Tan}^{-1} u + \text{Tan}^{-1}(1/u) = \frac{\pi}{2}$, if $u > 0$.
73. Prove that $\text{Cos}^{-1} u + \text{Cos}^{-1}(-u) = \pi$, if $-1 \leqq u \leqq 1$.
74. Prove that $\text{Tan}^{-1} u - \text{Tan}^{-1} v = \text{Tan}^{-1}\frac{u-v}{1+uv}$, if $u > 0$ and $v > 0$.

9 Linear Equations and Graphs

9–1. SOLUTIONS OF SIMULTANEOUS EQUATIONS

Often problems arise that involve two or more unknowns and as many equations. The solution of such a problem requires the determination of numbers which simultaneously satisfy the given equations. Equations for which we seek common solutions are referred to as a *system of simultaneous equations*. If the system has at least one solution, it is said to be *consistent;* otherwise, it is called *inconsistent.*

Let us investigate the following pair of simultaneous linear equations:

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases} \tag{9–1}$$

It is desired to find all pairs of values of x and y which satisfy both equations, excluding from consideration the cases $a_1 = b_1 = 0$ and $a_2 = b_2 = 0$.

We proceed by multiplying both sides of the first equation by b_2 and both sides of the second by b_1, obtaining

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2, \\ a_2b_1x + b_2b_1y &= c_2b_1. \end{aligned}$$

Subtracting the second equation of this pair from the first, we obtain

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1. \tag{9–2}$$

If $(a_1b_2 - a_2b_1) \neq 0$, we find that

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}.$$

Thus, we see that we have exactly one value for x in any solution which may exist.

By multiplying the original equations by a_2 and a_1, respectively, and subtracting the first equation thus obtained from the second one, we obtain

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1. \tag{9–3}$$

If $(a_1b_2 - a_2b_1) \neq 0$, we find that

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Again, we have exactly one value for y in any solution which may exist.

Hence, if $(a_1b_2 - a_2b_1)$ is not zero, we have at most one solution of the pair of given equations, namely,

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{9–4}$$

Substitution of these values for x and y in (9–1) shows that we really have a solution. The reader should verify this solution by making the substitutions.

Consistent and Independent Equations. If $(a_1b_2 - a_2b_1) \neq 0$, we have exactly one solution of (9–1), which is expressed by (9–4), and the given equations are called *consistent and independent.* We may consider, as an example, the pair of equations

$$\begin{cases} 2x + 3y = 24, \\ 5x - 2y = 22. \end{cases}$$

Here $a_1b_2 - a_2b_1 = (2)(-2) - (5)(3) = -19$, and the values $x = 6$ and $y = 4$ give a solution. In other words, when these values are substituted in the two given equations, both equations are satisfied. Thus,

$$\begin{cases} 2(6) + 3(4) = 24, \\ 5(6) - 2(4) = 22. \end{cases}$$

Cases with $a_1b_2 - a_2b_1 = 0$. Let us now consider cases in which $a_1b_2 - a_2b_1 = 0$. If $a_1 \neq 0$, division by a_1 gives

$$b_2 = \frac{a_2}{a_1} b_1.$$

We then define $k = \dfrac{a_2}{a_1}$, and we have

$$a_2 = ka_1, \quad b_2 = kb_1. \tag{9–5}$$

Now let $a_1 = 0$. Then, since b_1 cannot be zero by our initial assumption, and since $a_2b_1 = 0$, we have $a_2 = 0$. In this case, if we define $k = \dfrac{b_2}{b_1}$, we see that (9–5) again follows. Thus, $a_1b_2 - a_2b_1 = 0$ has

been shown to mean that the left members of (9–1) are proportional.

We note also that if $a_1b_2 - a_2b_1 = 0$, equations (9–2) and (9–3) become

$$\text{(9–6)} \qquad \begin{cases} 0 \cdot x = c_1b_2 - c_2b_1, \\ 0 \cdot y = a_1c_2 - a_2c_1. \end{cases}$$

It is clear that these equations cannot be satisfied by any pair of values of x and y, unless both right members are also zero. Accordingly, if we wish to determine whether or not the given system has a solution when $a_1b_2 - a_2b_1 = 0$, it becomes necessary to take into account the two cases of the right sides of (9–6) being zero or not zero.

Consistent and Dependent Equations. In the first case referred to in the preceding paragraph, $a_1b_2 - a_2b_1 = 0$, and $c_1b_2 - c_2b_1$ and $a_1c_2 - a_2c_1$ are also equal to zero. Hence, substituting the values of b_2 and a_2 from (9–5), we have

$$(c_1k - c_2)b_1 = (c_1k - c_2)a_1 = 0.$$

Since a_1, b_1 are not both zero, it follows that $c_2 = kc_1$. This means that one of the original equations is a constant multiple of the other, and any pair of values of x and y that satisfy one equation will also satisfy the other. Under this condition, the equations are said to be *consistent and dependent.*

We have as an illustrative example the equations

$$\begin{cases} 2x + 3y = 24, \\ 4x + 6y = 48. \end{cases}$$

Here $a_1b_2 - a_2b_1 = (2)(6) - (4)(3) = 0$. Also, the coefficients of the unknowns and the constant term in the second equation are multiples of the coefficients of the unknowns and the constant term in the first, and the multiplier is 2 for all three terms. Infinitely many solutions exist. Some of them are: $x = 0, y = 8$; $x = 12, y = 0$; and $x = t, y = \dfrac{24 - 2t}{3}$, where t is any number.

Inconsistent Equations. Finally, let us suppose that in the original equations $a_1b_2 - a_2b_1 = 0$ and at least one of the numbers $c_1b_2 - c_2b_1, a_1c_2 - a_2c_1$ is different from zero. Hence, no solution of (9–1) exists and the equations are *inconsistent.* This case is characterized by the condition that one of the numbers $(c_1k - c_2)b_1$, $(c_1k - c_2)a_1$ is not zero, in view of (9–5). It follows that $c_2 \neq kc_1$.

Hence, the multiplier for the left members of (9–1) does not apply to the constant terms.

Consider, for example, the equations

$$\begin{cases} 2x + 3y = 24, \\ 4x + 6y = 7. \end{cases}$$

Here the coefficients of the unknowns in the second equation are multiples of those in the first equation, and the multiplier is 2 for both terms; however, the multiplier for the constants is not the same as that for the other terms. Hence, these equations are inconsistent.

9–2. ALGEBRAIC SOLUTION OF LINEAR EQUATIONS IN TWO UNKNOWNS

To solve a consistent and independent system of two linear equations in two unknowns, we reduce the system to one equation in one unknown by eliminating one of the unknowns. The following example will illustrate two commonly used methods for eliminating the unknowns.

Example 9–1. Solve the equations

$$\begin{cases} 2x + 3y = 24, \\ 5x - 2y = 22. \end{cases}$$

Solution: Since $a_1b_2 - a_2b_1 = (2)(-2) - (5)(3) = -19$, the equations are consistent and independent, and there is but one solution. If we use the method of elimination by addition or subtraction, the procedure is the same as that indicated in obtaining the solution (9–4) from equations (9–1).

To eliminate y, multiply the first equation by 2 and the second by 3, in order to make the coefficients of y numerically equal in both equations. We thus obtain

$$\begin{cases} 4x + 6y = 48, \\ 15x - 6y = 66. \end{cases}$$

Adding, we get $$19x = 114.$$

Solving for x, we have $$x = 6.$$

Now, substitute 6 for x in the first of the original equations. Then

$$3y = 24 - 2x = 24 - 12 = 12,$$

or

$$y = 4.$$

Alternate Solution: If we use the method of elimination by substitution, we begin by solving the first equation for y in terms of x. We thus get

$$y = \frac{24 - 2x}{3}.$$

We then substitute $\frac{24 - 2x}{3}$ for y in the second equation and obtain

$$5x - 2\left(\frac{24 - 2x}{3}\right) = 22.$$

Solving for x, we have

$$15x - 2(24 - 2x) = 66,$$

or

$$15x - 48 + 4x = 66.$$

Hence,

$$19x = 114,$$

and

$$x = 6.$$

Substituting 6 for x, as before, we find that $y = 4$.

Example 9–2. A grocer has some coffee selling at 80 cents per pound and some at 90 cents per pound. How much of each must he use to get a mixture of 100 pounds worth 86 cents per pound?

Solution: Let x = number of pounds of 80-cent coffee, and y = number of pounds of 90-cent coffee.
Then

$$x + y = 100,$$

and

$$0.80x + 0.90y = 0.86\,(100).$$

Simplifying, we have

$$\begin{cases} x + y = 100, \\ 8x + 9y = 860. \end{cases}$$

These equations have the single solution $x = 40$, $y = 60$.

9–3. LINEAR EQUATIONS IN THREE UNKNOWNS

In the solution of a system of three equations in three unknowns, one method is to employ the following steps, which we do not justify here:

1. Eliminate one of the unknowns from a pair of the equations; then eliminate this same unknown from another pair of the original equations.
2. Solve the resulting two equations for the two remaining unknowns.
3. Substitute the values found in step 2 in any one of the original equations to find the third unknown.

Example 9–3. Solve the system of equations

$$\begin{cases} 2x + 3y - z = 5, \\ x - 5y + 2z = 1, \\ 3x + y - 4z = -1. \end{cases}$$

Solution: Eliminate z from the first and second equations by addition to obtain

$$5x + y = 11.$$

Now eliminate z from the second and third given equations by addition to obtain

$$5x - 9y = 1.$$

We then consider the equations

$$\begin{cases} 5x + y = 11, \\ 5x - 9y = 1. \end{cases}$$

Solving these equations for x and y by subtraction, we have $x = 2$ and $y = 1$. Substitution of these values in the first given equation gives $z = 2$. Hence, the solution of the given system is $x = 2$, $y = 1$, $z = 2$.

EXERCISE 9–1

In each of the problems from 1 to 30, solve the given system of equations. Check all solutions.

1. $\begin{cases} 3x - 2y = 6, \\ x - 3y = 4. \end{cases}$ **2.** $\begin{cases} 3x - y = 7, \\ 2x + y = 8. \end{cases}$ **3.** $\begin{cases} y - 3x = 6, \\ x + 2y = 3. \end{cases}$

4. $\begin{cases} 3x + 2y = 1, \\ x - 2y = 5. \end{cases}$ **5.** $\begin{cases} 2x + 3y + 1 = 0, \\ 3x - y + 7 = 0. \end{cases}$ **6.** $\begin{cases} 2x + 2y - 3 = 0, \\ 5x + 3y + 4 = 0. \end{cases}$

7. $\begin{cases} 3x + 2y = 4, \\ 2x - 3y = 3. \end{cases}$ **8.** $\begin{cases} x + 2y = 3, \\ 2x + 3y = 1. \end{cases}$ **9.** $\begin{cases} 3x + y + 7 = 0, \\ 4x + 8y + 9 = 0. \end{cases}$

10. $\begin{cases} x + 2y + 1 = 0, \\ x - 4y + 2 = 0. \end{cases}$ **11.** $\begin{cases} 2x + 3y - 1 = 0, \\ 3x + y + 3 = 0. \end{cases}$ **12.** $\begin{cases} 2x + 6y - 7 = 0, \\ 3x - 8y + 9 = 0. \end{cases}$

13. $\begin{cases} -x + \frac{1}{2}y = \frac{3}{5}, \\ -\frac{2}{3}x + \frac{3}{4}y = \frac{7}{9}. \end{cases}$ **14.** $\begin{cases} x - \frac{1}{2}y = 3, \\ \frac{3}{2}x + \frac{1}{3}y = \frac{1}{2}. \end{cases}$ **15.** $\begin{cases} 2x + 4y - 5z = 3, \\ 3x + y - 7z = 2, \\ 4x + 8y - 10z = 1. \end{cases}$

16. $\begin{cases} 3x - y + 2z = 4, \\ x + 2y - 3z = 1, \\ 2x - 3y + z = 2. \end{cases}$ **17.** $\begin{cases} x + 2y - z = 3, \\ -x + 4y + 2z = 1, \\ 3x + y - 3z = 2. \end{cases}$

18. $\begin{cases} x + 2y + 3z + 1 = 0, \\ x - 4y + 5z + 1 = 0, \\ 2x + 6y + 7z + 2 = 0. \end{cases}$ **19.** $\begin{cases} -3x + 8y + 9z - 3 = 0, \\ 2x + 3y + z - 4 = 0, \\ 3x - 2y - 2z - 4 = 0. \end{cases}$

20. $\begin{cases} x + 2y - 3z + 1 = 0, \\ -x + 3y - 4z - 5 = 0, \\ 2x + 6y - 4z + 3 = 0. \end{cases}$ **21.** $\begin{cases} 3x - y + 5z = 0, \\ x - 4z = 2, \\ 4x - 2y - 3z + 1 = 0. \end{cases}$

22. $\begin{cases} 3x + 2y + z = 2, \\ 5x + y + 3 = 0, \\ 2x - 3y - 4z + 5 = 0. \end{cases}$ **23.** $\begin{cases} 3x - 4y + 2z = 3, \\ 2x + y = 1, \\ 5x - 3y + 4z + 5 = 0. \end{cases}$

24. $\begin{cases} 3x - y + 4z = 2, \\ 4x + 4y + 4z = 5, \\ 2x - y + 6z = 9. \end{cases}$ **25.** $\begin{cases} 2x + 3y - 5z + 2 = 0, \\ 5y + 3z - 2 = 0, \\ x - y - \frac{1}{2}z - \frac{3}{2} = 0. \end{cases}$

26. $\begin{cases} \quad\;\; 3y + 2z = 4, \\ 2x - 2y + 3z = 3, \\ 3x \qquad\; + 4z = 2. \end{cases}$

27. $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} - \dfrac{1}{z} = 2, \\ \dfrac{1}{x} - \dfrac{4}{y} + \dfrac{2}{z} = 0, \\ \dfrac{1}{2x} + \dfrac{2}{y} + \dfrac{1}{z} = 1. \end{cases}$

28. $\begin{cases} \dfrac{4}{x} + \dfrac{2}{y} - \dfrac{2}{z} = \dfrac{1}{3}, \\ \dfrac{1}{x} - \dfrac{2}{y} + \dfrac{1}{z} = \dfrac{1}{3}, \\ \dfrac{2}{x} - \dfrac{1}{y} - \dfrac{3}{z} = 0. \end{cases}$

29. $\begin{cases} 3(x + y) - 4(x - y) = 5, \\ 4(x + y) - 3(x - y) = 5. \end{cases}$

30. $\begin{cases} 3(x + y) - 4(x - y) = 5, \\ 4(x - y) - 3(x + y) = 7. \end{cases}$

For each of the following systems of equations, determine whether it is consistent and independent, consistent and dependent, or inconsistent.

31. $\begin{cases} 2x - 3y = -5, \\ 4x - 6y = -3. \end{cases}$

32. $\begin{cases} 3x - 5y + 8 = 0, \\ x + 8y - 10 = 0. \end{cases}$

33. $\begin{cases} 3x - 2y = 8, \\ 6x - 4y = 8. \end{cases}$

34. $\begin{cases} 2x + 4y = 3, \\ x + 2y = 6. \end{cases}$

35. $\begin{cases} 2x - 7y + 1 = 0, \\ 21y - 6x - 3 = 0. \end{cases}$

36. $\begin{cases} 45x - 27y = 21, \\ 15x - 9y = 7. \end{cases}$

37. $\begin{cases} 2x - 3y = 1, \\ 4x + 6y = 2. \end{cases}$

38. $\begin{cases} -9x + 12y = 3, \\ 3x + 4y = 1. \end{cases}$

39. $\begin{cases} 4x - 3y = 6, \\ 15y - 20x + 6 = 0. \end{cases}$

40. $\begin{cases} -x + 3y = 2, \\ 2x = 6y + 14. \end{cases}$

41. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = c, \\ ay + bx = ab. \end{cases}$

42. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = c, \\ bx + ay = abc. \end{cases}$

9–4. GRAPHS OF LINEAR FUNCTIONS

The discussion of rectangular coordinates in Section 2–1 set the stage for the pictorial representation of a function. By this representation of a function $f(x)$, we mean the graph of the equation $y = f(x)$. It consists of all points, and only those points, whose coordinates x and y satisfy the equation.

In the same section we considered the graphing of lines parallel to the coordinate axes. The equation $x = 3$ was shown to represent a vertical line, that is, a line parallel to the y-axis which intersects the x-axis at the point $(3, 0)$. This line thus includes all points 3 units to the right of the y-axis. This example illustrates the fact that a linear equation in x alone represents a line that is parallel to the y-axis. Similarly, $y = 2$ was shown to be the equation of the horizontal line which is parallel to and 2 units above the x-axis. And this example illustrates the fact that an equation in y alone represents a line parallel to the x-axis. Furthermore, it is proved in analytic geometry that the graph of every first-degree, or linear, equation in x and y is a straight line; and, conversely, that every straight line is the graph of a linear equation.

We shall proceed by first preparing a table of corresponding values in a given problem and then plotting the corresponding points on the coordinate system to obtain the graph of the equation. The following illustrative examples will point the way toward an understanding of the procedure in the graphing of linear equations.

Example 9–4. Graph the function $2x + 3$.

Solution: Let $y = 2x + 3$. Then assign any values for x, substitute them in the equation, and obtain the corresponding values for y. The table and the graph are shown in Fig. 9–1.

Since a straight line is definitely determined when two points are known, only two pairs of values of x and y are needed in graphing a linear equation. We can, however, use three points in order to check our work.

x	y
0	3
1	5
−2	−1

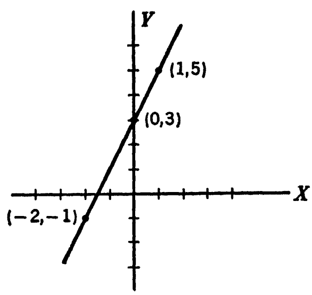

FIG. 9–1.

x	y
−3	4
0	2
3	0

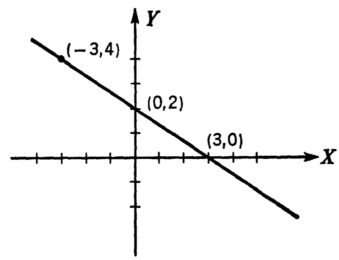

FIG. 9–2.

Example 9–5. Graph the equation $2x + 3y = 6$.

Solution: The equation may be solved for y in terms of x. Then

$$y = -\frac{2}{3}x + 2.$$

A table of corresponding values of x and y and the graph are shown in Fig. 9–2.

9–5. INTERCEPTS

In general, the points where a curve crosses the coordinate axes are the easiest to obtain.

The x-intercepts are the values of x at the points where the graph crosses the x-axis. Since $y = 0$ on this axis, the x-intercepts are the values of x that correspond to $y = 0$. Similarly, the y-intercepts are the values of y at which the graph crosses the y-axis. They are the values of y that correspond to $x = 0$. Hence, we have the following rule:

To find the x-intercepts, set $y = 0$ in the equation and solve for x. To find the y-intercepts, set $x = 0$ in the equation and solve for y.

Example 9–6. Find the intercepts of the line

$$2x + 3y = 6.$$

Solution: To find the x-intercept, let $y = 0$. Then $2x = 6$, and $x = 3$.

To find the y-intercept, let $x = 0$. Then $3y = 6$, and $y = 2$.

Note that the intercepts $x = 3$ and $y = 2$ found in this solution correspond to the points $(3, 0)$ and $(0, 2)$, respectively, where the line in Fig. 9–2 crosses the coordinate axes.

9–6. GRAPHICAL SOLUTION OF LINEAR EQUATIONS IN TWO UNKNOWNS

In the graphical solution of two linear equations in two unknowns, the graphs of the two equations are drawn with reference to the same coordinate axes. Since the solution of two equations in x and y is a pair of values of x and y which satisfy both equations, the solution must represent graphically a point common to both lines represented by the equations. Hence, the values of x and y which satisfy both equations give the coordinates of the point of intersection of the lines. We find, therefore, that the two lines intersect in a single point, are parallel, or are identical, according as the equations are consistent and independent, inconsistent, or consistent and dependent.

Example 9–7. Solve graphically

$$\begin{cases} 2x + 3y = 24, \\ 5x - 2y = 22. \end{cases}$$

Solution: Tables of corresponding values for the two equations and also their graphs are shown in Fig. 9–3.

It is seen from the graphs that the lines intersect at the point $(6, 4)$. That $x = 6$, $y = 4$ gives the solution of the given equations may be checked by substitution.

$2x + 3y = 24$

x	y
0	8
12	0

$5x - 2y = 22$

x	y
0	-11
$\frac{22}{5}$	0

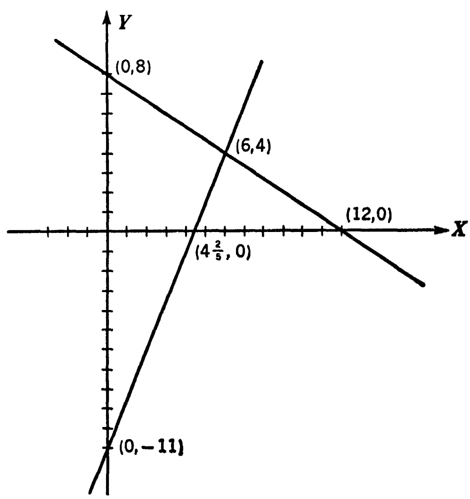

FIG. 9–3.

EXERCISE 9–2

In each of problems from 1 to 9, graph the given function. In each case give the x- and y-intercepts.

1. $4x - 3y = 1.$ **2.** $y = 2x - 8.$ **3.** $y = x - 4.$

4. $y = x + 5.$ **5.** $y = 3x.$ **6.** $y = x.$

7. $y = 3x + 4.$ **8.** $y = x - 3.$ **9.** $y = 3x - 5.$

Solve each of the following systems of equations graphically.

10. $\begin{cases} x - 3y = 1, \\ 3x - 2y = 0. \end{cases}$ **11.** $\begin{cases} 3y - 2x = 0, \\ x + y = 2. \end{cases}$ **12.** $\begin{cases} 2y = x - 3, \\ 2x = y + 3. \end{cases}$

13. $\begin{cases} 2y = x + 3, \\ 2x = 16 - 3y. \end{cases}$ **14.** $\begin{cases} 3x - y = 4, \\ y - 3x = 1. \end{cases}$ **15.** $\begin{cases} \dfrac{3x}{x - y} = 8, \\ \dfrac{3}{2(x - y)} = 4. \end{cases}$

16. $\begin{cases} 2x - y = 4, \\ 3x + 2y = 12. \end{cases}$ **17.** $\begin{cases} 2x + 4 = 5y - 3, \\ 3y - 4 = 4x + 2. \end{cases}$ **18.** $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 3, \\ 2x - 3y = 12. \end{cases}$

10 Determinants

10–1. DETERMINANTS OF THE SECOND ORDER

Let us consider the following system of two linear equations in two unknowns:

$$(10\text{–}1) \qquad \begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases}$$

The solution of these equations by the method of Section 9–1 is given by

$$(10\text{–}2) \qquad x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

It is understood that $a_1b_2 - a_2b_1 \neq 0$.

We shall at this point introduce a more convenient way of writing the expression $a_1b_2 - a_2b_1$. The notation which we select for this purpose will enable us to express also the numerators of (10–2) by means of the same symbol with the proper changes in letters.

In choosing a symbol we shall select a form which will exhibit the numbers a_1, b_1, a_2, b_2 in the same relative positions as in (10–1). Thus, we write

$$\begin{matrix} a_1 & b_1 \\ a_2 & b_2. \end{matrix}$$

This arrangement of the four numbers in a square array, consisting of two rows and two columns, is then enclosed within vertical bars, as follows:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

This symbol represents a *determinant of second order.*

Thus, we start with a square array, or *matrix,* such as

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

We then associate with this array a number $a_1b_2 - a_2b_1$, called its *determinant,* which is denoted in the following manner:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The numbers a_1, b_1, a_2, b_2 are called *elements* of the determinant. The numbers a_1 and b_2 lie on the *principal diagonal*; the numbers a_2 and b_1 lie on the *secondary diagonal.*

Note. We observe that the "expanded" value of the foregoing determinant is equal to the product of the elements on the principal diagonal minus the product of the elements on the secondary diagonal. It is interesting to note also that this value is the algebraic sum of all possible products obtainable by taking one and only one element from each row and one element from each column. Each product is preceded by a plus sign or a minus sign, according to a rule to be stated in Section 10–2.

Using the notation of determinants, we can write the solution (10–2) of (10–1) in the form

$$(10\text{–}3) \qquad x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

We note that the value of each of the unknowns in (10–3) may be written as a fraction whose denominator is the determinant of the coefficients as they stand in (10–1), and whose numerator is the determinant formed from that of the denominator by replacing the column of coefficients of the unknown in question by the column of constant terms.

Note. If $a_2 = ka_1$ and $b_2 = kb_1$, where k is any number, then

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

In this case, the equations of the system (10–1) are inconsistent unless both numerators of the fractions in (10–2) are also equal to zero, that is, unless

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_1 \end{vmatrix} = 0.$$

Therefore, the equations of the system (10–1) represent distinct, parallel straight lines if $c_2 \neq kc_1$, or they represent coincident lines if $c_2 = kc_1$.

Example 10–1. Solve the system of equations

$$\begin{cases} 4x + 3y = 1, \\ 3x - 2y = 22. \end{cases}$$

Solution: Using determinants, we have

$$x = \frac{\begin{vmatrix} 1 & 3 \\ 22 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{-2-66}{-8-\ 9} = \frac{-68}{-17} = 4.$$

Also,

$$y = \frac{\begin{vmatrix} 4 & 1 \\ 3 & 22 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{88-3}{-8-9} = \frac{85}{-17} = -5.$$

10–2. DETERMINANTS OF THE THIRD ORDER

A determinant of the third order is a number designated by a square array of nine elements arranged in three rows and three columns and enclosed within vertical bars. An example is

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \tag{10–4}$$

The value, or expansion, of the determinant (10–4) is defined as the quantity

$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \tag{10–5}$$

or as the quantity

$$D = a_1b_2c_3 - a_1b_3c_2 + b_1c_2a_3 - b_1a_2c_3 + c_1a_2b_3 - c_1b_2a_3, \tag{10–6}$$

Here the products such as $a_1b_2c_3$, $a_1b_3c_2$, and $b_1a_2c_3$ are known as the *terms* of the determinant.

Minors and Cofactors. In any determinant the *minor* of a given element is the determinant of the array which remains after deleting all the elements that lie in the same row and in the same column as the given element. Thus, in (10–4) the minors of a_1, b_1, c_1 are, respectively,

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

The *cofactor* of an element which lies in the ith row and kth column is equal to the minor of that element if $i + k$ is even, and is equal to the negative of the minor if $i + k$ is odd. That is,

$$\text{cofactor} = (-1)^{i+k} \cdot \text{minor}.$$

Thus, in the determinant in (10–4), the cofactor of a_1 equals the minor of a_1, since a_1 lies in the first row and in the first column and $i + k = 1 + 1 = 2$, which is even. Similarly, the cofactor of b_1 is the negative of the minor of b_1, since $i + k = 1 + 2 = 3$, which is odd.

Often the following procedure may prove more convenient for finding the sign corresponding to a given element. Beginning with + in the upper left-hand corner, change sign from place to place, moving horizontally or vertically, until the position for the element in question is reached. The schematic arrangement of signs corresponding to the elements of a third-order determinant is thus as follows:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Note that the sign for any position is independent of the path followed in arriving at that position.

We shall designate the value of the cofactor of an element by the corresponding capital letter, and we shall use the subscript that occurs with the element itself. Thus, the cofactors of a_1, b_1, c_1 are, respectively,

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}. \tag{10–7}$$

Hence, (10–5) for the expansion of the determinant may also be written as follows:

$$D = a_1A_1 + b_1B_1 + c_1C_1. \tag{10–8}$$

This sum is called the *expansion of the determinant according to the elements of the first row.*

We observe at this point that the right member of (10-6) represents all possible products, here 3! in number, that can be formed from the determinant in (10–4) by taking one and only one element from each row and each column. It follows also that the value of the determinant is the same, regardless of the row or column

according to which the expansion is made. Thus, we may express the determinant as

(10–9) $$D = b_1B_1 + b_2B_2 + b_3B_3,$$

or as

(10–10) $$D = a_3A_3 + b_3B_3 + c_3C_3.$$

These equations represent the expansion according to the elements of the second column and according to the elements of the third row, respectively.

Example 10–2. Expand the determinant

$$\begin{vmatrix} 2 & -5 & 3 \\ 6 & 2 & 1 \\ -1 & 7 & 4 \end{vmatrix}$$

according to the elements of the first row and according to the elements of the third column.

Solution: The expansion according to the elements of the first row is

$$2\begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix} + 5\begin{vmatrix} 6 & 1 \\ -1 & 4 \end{vmatrix} + 3\begin{vmatrix} 6 & 2 \\ -1 & 7 \end{vmatrix}.$$

This reduces to

$$2(8 - 7) + 5(24 + 1) + 3(42 + 2) = 259.$$

Expanding according to the elements of the third column, we have

$$3\begin{vmatrix} 6 & 2 \\ -1 & 7 \end{vmatrix} - 1\cdot\begin{vmatrix} 2 & -5 \\ -1 & 7 \end{vmatrix} + 4\begin{vmatrix} 2 & -5 \\ 6 & 2 \end{vmatrix} = 3(42 + 2) - (14 - 5) + 4(4 + 30) = 259.$$

10–3. PROPERTIES OF DETERMINANTS

From the definition of the value of a determinant we may deduce the following important properties of determinants. These properties supply us with more convenient methods for evaluating a determinant.

Note. For a more complete discussion of these properties, the student is referred to any one of the various treatises on determinants or to texts on the theory of equations or on solid analytic geometry, where he will also find proofs which apply to determinants of any order.

The properties listed here will be employed in examples that follow, and their usefulness in simplifying determinants will be illustrated.

Property 1. The value of a determinant is not changed if its rows and columns are interchanged.

Property 2. If all the elements of a row, or of a column, are multiplied by the same number, the value of the determinant is multiplied by that number. For example,

$$\begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Property 3. If two rows, or two columns, of a determinant are identical or proportional, the value of the determinant is zero. For example, let the first two columns be identical, as in the determinant

$$D = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}.$$

Then, expanding according to the elements of the third column, we have

$$D = c_1 \begin{vmatrix} a_2 & a_2 \\ a_3 & a_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_1 \\ a_3 & a_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0.$$

Property 4. The value of a determinant is not changed if we add to the elements of any column (row) any arbitrary multiple of the elements of any other given column (row).

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + nb_1 & b_1 & c_1 \\ a_2 + nb_2 & b_2 & c_2 \\ a_3 + nb_3 & b_3 & c_3 \end{vmatrix}.$$

The proof follows. Expanding according to the elements of the first column, we find that

$$\begin{vmatrix} a_1 + nb_1 & b_1 & c_1 \\ a_2 + nb_2 & b_2 & c_2 \\ a_3 + nb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + n \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

The last determinant vanishes, since two columns are identical.

Example 10–3. Evaluate the determinant

$$D = \begin{vmatrix} 4 & 3 & -1 \\ 5 & 1 & 2 \\ 2 & 4 & 3 \end{vmatrix}.$$

Solution: By adding 2 times the elements of the first row to the elements of the second row, we obtain

$$\begin{vmatrix} 4 & 3 & -1 \\ 5 & 1 & 2 \\ 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 3 & -1 \\ 13 & 7 & 0 \\ 2 & 4 & 3 \end{vmatrix}.$$

If now we add 3 times the elements of the first row to the third row, the determinant becomes

$$\begin{vmatrix} 4 & 3 & -1 \\ 13 & 7 & 0 \\ 14 & 13 & 0 \end{vmatrix}.$$

Expanding according to the elements of the third column, we have

$$(-1)\begin{vmatrix} 13 & 7 \\ 14 & 13 \end{vmatrix} = -71.$$

In this example, we first converted the given determinant to one in which all but one of the elements of the third column are zero. For the final expansion, the given determinant was thus reduced to a determinant of the second order, and its value was easily found.

Example 10–4. Without expanding the determinant, show that $x = -2$ satisfies the equation

$$\begin{vmatrix} 3x^2 & x^3 & -x \\ 3 & 1 & 7 \\ 6 & -4 & 1 \end{vmatrix} = 0.$$

Solution: Substituting -2 for x in the determinant, we have

$$\begin{vmatrix} 12 & -8 & 2 \\ 3 & 1 & 7 \\ 6 & -4 & 1 \end{vmatrix}.$$

This equals zero, since the first and third rows are proportional. Hence, the equation is satisfied by $x = -2$.

EXERCISE 10–1

In each of the problems from 1 to 12, evaluate the given determinant.

1. $\begin{vmatrix} 2 & -5 \\ 3 & -5 \end{vmatrix}$. **2.** $\begin{vmatrix} 2 & 7 \\ 3 & -5 \end{vmatrix}$. **3.** $\begin{vmatrix} 7 & 2 \\ 14 & 4 \end{vmatrix}$. **4.** $\begin{vmatrix} 9 & 7 \\ 8 & 14 \end{vmatrix}$.

5. $\begin{vmatrix} 3 & -5 \\ 6 & -10 \end{vmatrix}$. **6.** $\begin{vmatrix} 3 & 8 \\ 6 & -6 \end{vmatrix}$. **7.** $\begin{vmatrix} 1 & -3 & 2 \\ 2 & 1 & 4 \\ -2 & 1 & -3 \end{vmatrix}$. **8.** $\begin{vmatrix} 0 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & 1 & 4 \end{vmatrix}$.

9. $\begin{vmatrix} 0 & 2 & 1 \\ -1 & 3 & 2 \\ 3 & 4 & -5 \end{vmatrix}$. **10.** $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$. **11.** $\begin{vmatrix} 2 & 4 & -5 \\ -6 & -1 & 4 \\ 4 & 8 & -9 \end{vmatrix}$. **12.** $\begin{vmatrix} 2 & 2 & 6 \\ 1 & -6 & 3 \\ 5 & 7 & 15 \end{vmatrix}$.

In each of the problems from 13 to 19, solve the given system of simultaneous equations by means of determinants. Check all solutions by substitution in the equations.

13. $\begin{cases} x + 3y = 5, \\ 2x - 4y = 7. \end{cases}$ **14.** $\begin{cases} 2x - y = 1, \\ 3x + 2y = 4. \end{cases}$ **15.** $\begin{cases} 3x + 2y = 5, \\ 2x - 3y = 5. \end{cases}$

16. If D represents the determinant in Problem 10, show that $D = 0$ is the equation of the straight line through the points (x_1, y_1) and (x_2, y_2).

17. Find the x-intercept and the y-intercept of the line whose equation is

$$\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 2 & -3 & 1 \end{vmatrix} = 0.$$

18. Solve graphically the following system of equations:

$$\begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 1/2 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} x & y & 1 \\ 1 & 0 & 1 \\ 0 & 1/2 & 1 \end{vmatrix} = 0.$$

19. Find the coordinates of the vertices of the triangle whose sides are the straight lines $x - y + 2 = 0$, $2x + 9y + 15 = 0$, and $7x + 4y - 30 = 0$.

20. Find the vertices of the parallelogram formed by the following lines:

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ -3 & 2 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} x & y & 1 \\ 6 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} x & y & 1 \\ 1 & 3 & 1 \\ 2 & 4 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} x & y & 1 \\ 5 & 1 & 1 \\ 1 & -3 & 1 \end{vmatrix} = 0.$$

10–4. SOLUTION OF THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

Let us consider the following system of linear equations:

$$(10\text{–}11)\qquad \begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

The determinant of the coefficients of the unknowns is

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

For the solution of such a system, we employ the following theorem, which is known is *Cramer's Rule.*

Theorem. If the determinant D of the coefficients of the system is not equal to zero, the system has just one solution. In this solution, the value of any unknown is equal to a fraction whose denominator is D and whose numerator is obtained from D by replacing the column of coefficients of the unknown in question by the column of constants d_1, d_2, and d_3.

Proof. Let the numerators of the fractions for x, y, z be denoted by D_1, D_2, D_3, respectively. We proceed to show that if equations (10–11) are to be satisfied, then

$$(10\text{–}12)\qquad Dx = D_1, \quad Dy = D_2, \quad Dz = D_3.$$

Specifically, the equation for x will have the form

$$(10\text{–}13)\qquad x \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

Similar equations may be written for y and z.

To find x, the first equation of the system in (10–11) is multiplied by the cofactor A_1, the second by A_2, and the third by A_3. After adding and collecting terms, we obtain

$$(10\text{–}14)\qquad \begin{aligned} &(a_1A_1 + a_2A_2 + a_3A_3)x + (b_1A_1 + b_2A_2 + b_3A_3)y \\ &\quad + (c_1A_1 + c_2A_2 + c_3A_3)z = d_1A_1 + d_2A_2 + d_3A_3. \end{aligned}$$

The coefficient of x in (10–14) is the expanded value of D according to the elements of the first column. The coefficient of y is

$$b_1A_1 + b_2A_2 + b_3A_3;$$

this is equal to

$$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix},$$

which equals zero, since two columns are identical. Similarly, the coefficient of z is zero. Hence, we have shown that the left side of (10–14) is the expanded form of the left side of (10–13) and that the two are therefore the same.

The right side of (10–14) is $d_1A_1 + d_2A_2 + d_3A_3$, which is the expansion of the determinant

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

This determinant may be obtained from D by replacing the coefficients of x by the column of constants; that is, it is the expansion of the determinant that we have called D_1. Hence, the equation $Dx = D_1$ in (10–12) is established. By a similar procedure we can show that the equations $Dy = D_2$ and $Dz = D_3$ are also valid.

If $D \neq 0$, the value of x is given by the equation

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \tag{10–15}$$

Similar values may be found for y and z. The proof is complete.

The proof of (10–12) is valid whether $D \neq 0$ or $D = 0$. If $D \neq 0$, the equations of the system (10–11) are consistent and have only one solution, which is of the form (10–15); that is,

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}. \tag{10–16}$$

If $D = 0$, and any one or more of the other determinants, D_1, D_2, D_3, is not zero, the given system of equations has no solution and is inconsistent.

If $D = 0$, and all the other determinants are zero, the equations of the system (10–11) may be consistent or inconsistent. If they are consistent, there are infinitely many solutions. This case will be treated in Section 10–5.

The following example will illustrate the case for which $D \neq 0$.

Example 10–5. Solve the system of equations

$$\begin{cases} x - y + z = 1, \\ x + y - 2z = 3, \\ 2x - y + 3z = 4. \end{cases}$$

Solution: Here

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix} = 5; \quad D_1 = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & -2 \\ 4 & -1 & 3 \end{vmatrix} = 11;$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & -2 \\ 2 & 4 & 3 \end{vmatrix} = 8; \quad D_3 = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 3 \\ 2 & -1 & 4 \end{vmatrix} = 2.$$

Hence, $x = \dfrac{D_1}{D} = \dfrac{11}{5}$, $y = \dfrac{D_2}{D} = \dfrac{8}{5}$, $z = \dfrac{D_3}{D} = \dfrac{2}{5}$. If we check by substitution, we find that these values satisfy the given equations.

10–5. SYSTEMS OF THREE LINEAR EQUATIONS IN THREE UNKNOWNS WHEN $D = 0$

We note that when $D = 0$, the system (10–11) will not have a solution if any one of the other determinants D_1, D_2, D_3 is different from zero. Suppose that a solution is given by

$$x = r, \quad y = s, \quad z = t.$$

Then the equations (10–12) become

$$r \cdot 0 = D_1, \quad s \cdot 0 = D_2, \quad t \cdot 0 = D_3.$$

It follows that $D_1 = 0$, $D_2 = 0$, $D_3 = 0$.

The following example will illustrate the case of consistent equations where $D = 0$ and $D_1 = D_2 = D_3 = 0$. The equations are said to be *dependent*, and they have infinitely many solutions. The student should construct an example to show that the equations (10–11) may be inconsistent when $D = 0$, even though $D_1 = D_2 = D_3 = 0$.

Example 10–6. Solve the system of equations

$$\begin{cases} x + y - z = 3, \\ 2x - y + 3z = 1, \\ 4x - 2y + 6z = 2. \end{cases}$$

Solution: Here

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 4 & -2 & 6 \end{vmatrix}.$$

This equals zero because the second and third rows are proportional. But, we also find that

$$D_1 = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -1 & 3 \\ 2 & -2 & 6 \end{vmatrix} = 0, \quad D_2 = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 3 \\ 4 & 2 & 6 \end{vmatrix} = 0, \quad D_3 = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 4 & -2 & 2 \end{vmatrix} = 0.$$

In this case the given equations have a solution. In fact, the second and third equations are proportional, and so either of these can be solved together with the first equation for two of the unknowns in terms of the third. For example,

$$x = \frac{4 - 2z}{3}, \quad y = \frac{5 + 5z}{3}.$$

Thus, we have a single value of x and a single value of y for every value of z. However, there are infinitely many values of z, and therefore infinitely many solutions of the given equations exist.

10–6. HOMOGENEOUS EQUATIONS

The system (10–11) is *homogeneous* if $d_1 = 0$, $d_2 = 0$, and $d_3 = 0$. Such a system always has the trivial solution $x = y = z = 0$. When $d_1 = d_2 = d_3 = 0$, it is seen that $D_1 = D_2 = D_3 = 0$, for each of these determinants has zero for every element in one column. If $D \neq 0$, it follows from (10–16) that we can have but one solution, which is given by

$$x = \frac{D_1}{D} = 0, \quad y = \frac{D_2}{D} = 0, \quad z = \frac{D_3}{D} = 0.$$

Hence, if the given system is to have a solution besides the trivial solution, D must equal zero. It may be shown that, if $D = 0$, nontrivial solutions always exist.

Example 10–7. Solve the system of equations

$$\begin{cases} x - y + z = 0, \\ 2x - 3y + 4z = 0, \\ 5x - 2y - z = 0. \end{cases}$$

Solution: The determinant D is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & 4 \\ 5 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ -1 & -3 & 1 \\ 3 & -2 & -3 \end{vmatrix} = 0.$$

Therefore, nontrivial solutions exist. To find these solutions, we proceed as follows:

Transpose z in each of the first two equations, and solve for x and y in terms of z. Then we have

$$D = \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} = -1, \quad D_1 = \begin{vmatrix} -z & -1 \\ -4z & -3 \end{vmatrix} = -z, \quad D_2 = \begin{vmatrix} 1 & -z \\ 2 & -4z \end{vmatrix} = -2z.$$

Hence,

$$x = \frac{D_1}{D} = z, \quad \text{and} \quad y = \frac{D_2}{D} = 2z.$$

Substitution shows that these values also satisfy the third equation. The given system therefore has infinitely many solutions, and the values of x, y, and z are related by the equations $x = z$ and $y = 2z$.

EXERCISE 10–2

In each of the problems from 1 to 6, solve the given system of simultaneous equations by means of determinants. Check all solutions by substitution in the equations.

1. $\begin{cases} 3x + y - z = 11, \\ x + 3y - z = 13, \\ x + y - 3z = 11. \end{cases}$

2. $\begin{cases} x - y - 2z = -1, \\ 5x - 2y = 0, \\ 12x - 4y + z = 3. \end{cases}$

3. $\begin{cases} x - y + 6z = 7, \\ 2x + 3y + 6z = 0, \\ x + 2y + 9z = 3. \end{cases}$

4. $\begin{cases} 2x + y + 3z = -10, \\ 2x - 2y + z = 2, \\ 6x + 2y - 2z = 5. \end{cases}$

5. $\begin{cases} 2x - y - 3z = 7, \\ x + 2y - z = 10, \\ 3x - 3y + 2z = -7. \end{cases}$

6. $\begin{cases} 2x - y = 3, \\ 2x - 3z = -1, \\ 3z - y = 2. \end{cases}$

For each of the following systems find at least one nontrivial solution, or show that there is no nontrivial solution.

7. $\begin{cases} x - 2y + 2z = 0, \\ 2x - 5y + z = 0, \\ 4x - 11y - z = 0. \end{cases}$

8. $\begin{cases} 2x - 3y + 4z = 0, \\ x + 3y - z = 0, \\ 7x + 3y + 5z = 0. \end{cases}$

9. $\begin{cases} x - 2y + 3z = 0, \\ 2x - y + 4z = 0, \\ 3x + y - z = 0. \end{cases}$

10–7. SUM AND PRODUCT OF DETERMINANTS

Closely related to Property 4 of Section 10–3 is a theorem concerning the sum of determinants. We shall illustrate the theorem for splitting the elements of a given column into two parts by means of the following equality for third-order determinants:

$$\begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

This can be shown to be true by expanding the three determinants according to the elements in the first column and noting that in the expansion the minors are the same for all three determinants.

Example 10–8. Show that

$$\begin{vmatrix} 2 & 4 & 5 \\ 1 & -1 & 0 \\ 3 & 7 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 5 \\ 4 & -1 & 0 \\ -1 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 5 \\ 5 & -1 & 0 \\ 2 & 7 & 6 \end{vmatrix}.$$

Solution: Expanding each of the determinants on the left side of the equation according to the elements in the first column, we have

$$2\begin{vmatrix} -1 & 0 \\ 7 & 6 \end{vmatrix} - \begin{vmatrix} 4 & 5 \\ 7 & 6 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 7 & 6 \end{vmatrix} - 4\begin{vmatrix} 4 & 5 \\ 7 & 6 \end{vmatrix} - \begin{vmatrix} 4 & 5 \\ -1 & 0 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 0 \\ 7 & 6 \end{vmatrix} - 5\begin{vmatrix} 4 & 5 \\ 7 & 6 \end{vmatrix} + 2\begin{vmatrix} 4 & 5 \\ -1 & 0 \end{vmatrix}.$$

This result is the expansion of the determinant on the right in the given equation according to the elements of the first column. Computing the value of each determinant in the given equation, we see that each side reduces to 47.

A similar theorem for splitting the elements of a given row is illustrated by the following example:

$$\begin{vmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & d_1 \\ a_2 & b_2 \end{vmatrix}.$$

This can be shown to be valid by expanding according to the elements of the first row.

Thus, consider the determinant

$$\begin{vmatrix} 4 & 5 \\ 1 & 7 \end{vmatrix}.$$

This may be written as a sum in various ways. Examples are

$$\begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 7 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 4 & 5 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 3 & 5 \end{vmatrix}.$$

We shall state without proof the rule for the product of two determinants of the same order. The product is equal to a determinant of like order in which the element of the ith row and kth column is the sum of the products of the elements of the ith row of the first determinant and the corresponding elements of the kth column of the second determinant. For example, for second-order determinants, we may write

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = \begin{vmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{vmatrix}.$$

That the product of the values of the two determinants on the left equals the value of the determinant on the right is checked by expanding. Thus, the desired equality becomes

$$(4 - 6) \cdot (40 - 42) = (950 - 946),$$

or

$$(-2) \cdot (-2) = 4.$$

To illustrate the multiplication of two third-order determinants, we have that

$$\begin{vmatrix} 3 & 2 & 5 \\ 1 & -1 & 2 \\ 4 & 6 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 7 & 6 \\ -3 & 9 & 3 \\ 5 & -4 & 2 \end{vmatrix}$$

is equal to

$$\begin{vmatrix} 3 \cdot 1 + 2(-3) + 5 \cdot 5 & 3 \cdot 7 + 2 \cdot 9 + 5(-4) & 3 \cdot 6 + 2 \cdot 3 + 5 \cdot 2 \\ 1 \cdot 1 + (-1)(-3) + 2 \cdot 5 & 1 \cdot 7 + (-1)9 + 2(-4) & 1 \cdot 6 + (-1)3 + 2 \cdot 2 \\ 4 \cdot 1 + 6(-3) + 0 \cdot 5 & 4 \cdot 7 + 6 \cdot 9 + 0(-4) & 4 \cdot 6 + 6 \cdot 3 + 0 \cdot 2 \end{vmatrix}.$$

This reduces to

$$\begin{vmatrix} 22 & 19 & 34 \\ 14 & -10 & 7 \\ -14 & 82 & 42 \end{vmatrix},$$

which is equal to -630. Also, computing the values of the given factors, we have $30(-21)$, which equals -630.

EXERCISE 10–3

In each of the first three problems, combine the given determinants into a single determinant, and evaluate the result.

1. $\begin{vmatrix} 0 & 2 & 2 \\ -3 & -1 & 7 \\ 4 & 8 & -10 \end{vmatrix} + \begin{vmatrix} 2 & 2 & -7 \\ -3 & -1 & 7 \\ 4 & 8 & -10 \end{vmatrix}.$

2. $\begin{vmatrix} 1 & 2 & 2 \\ 2 & -3 & 1 \\ 4 & 8 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 4 & 2 \\ 2 & 2 & 1 \\ 4 & 19 & -1 \end{vmatrix}.$

3. $\begin{vmatrix} 2 & 4 & -5 \\ -6 & -1 & 4 \\ 1 & 2 & 12 \end{vmatrix} + \begin{vmatrix} 2 & 4 & -5 \\ -6 & -1 & 4 \\ 4 & 3 & -6 \end{vmatrix} + \begin{vmatrix} 2 & 4 & -5 \\ -6 & -1 & 4 \\ -1 & 3 & -16 \end{vmatrix}.$

In each of the following problems, find the product of the determinants without evaluating the individual factors.

4. $\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$.

7. $\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} \cdot \begin{vmatrix} -1 & 4 \\ 3 & 10 \end{vmatrix}$.

8. $\left(\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix}\right) \cdot \begin{vmatrix} 2 & 6 \\ -3 & 8 \end{vmatrix}$.

9. $\left(\begin{vmatrix} 7 & -2 \\ 9 & 3 \end{vmatrix} \cdot \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix}\right) \cdot \left(\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} \cdot \begin{vmatrix} -3 & 2 \\ 1 & 4 \end{vmatrix}\right)$.

11 Complex Numbers

11-1. THE COMPLEX NUMBER SYSTEM

There are many problems that cannot be solved by the use of real numbers alone. We observe, for example, that the equation $x^2 + 1 = 0$ has no real root, since x^2 can never be negative if x is a real number. In order to provide solutions to such equations, a new system of numbers, called the *complex number system,* was introduced. Later in this book, we shall find many instances of solutions involving complex numbers.

We shall now define a *complex number* as an ordered pair of real numbers, which we denote by (a, b). If the numbers a and b are regarded as the Cartesian coordinates of a point in a rectangular coordinate system, we have a one-to-one correspondence between the set of complex numbers and the set of points in a plane. The plane is called the *complex plane.* Two complex numbers (a, b) and (c, d) are equal if and only if they correspond to the same point, that is, if and only if $a = c$ and $b = d$.

Addition, subtraction, and multiplication of complex numbers are defined as follows:

$$(a, b) + (c, d) = (a + c, b + d), \tag{11-1}$$

$$(a, b) - (c, d) = (a - c, b - d), \tag{11-2}$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc). \tag{11-3}$$

For example,

$$(1, 3) + (5, 2) = (6, 5),$$
$$(1, 3) - (5, 2) = (-4, 1),$$
$$(1, 3) \cdot (5, 2) = (-1, 17).$$

We also define the following special complex numbers:

$$\mathbf{0} = (0, 0), \tag{11-4}$$

$$\mathbf{1} = (1, 0), \tag{11-5}$$

$$i = (0, 1). \tag{11-6}$$

The complex number **0** serves as a zero of the complex number system, while **1** serves as a unit, in accordance with the following properties:

$$\mathbf{0} + (a, b) = (a, b) + \mathbf{0} = (a, b),$$
$$\mathbf{0} \cdot (a, b) = (a, b) \cdot \mathbf{0} = \mathbf{0},$$
$$\mathbf{1} \cdot (a, b) = (a, b) \cdot \mathbf{1} = (a, b).$$

The so-called *imaginary unit* $i = (0, 1)$ will be discussed in more detail in Section 11–2.

If k is a real number, we define

(11–7) $$k \cdot (a, b) = (k, 0) \cdot (a, b) = (ka, kb).$$

Also, we define

(11–8) $$-(a, b) = (-1) \cdot (a, b) = (-a, -b).$$

Since $(a, b) + (-a, -b) = (0, 0)$, the complex number $-(a, b)$ is called the *negative* of (a, b).

The Reciprocal of (a,b). If $(a, b) \neq \mathbf{0}$, then it has a reciprocal (x, y) such that

$$(a, b) \cdot (x, y) = \mathbf{1}.$$

Furthermore, the reciprocal is given by

(11–9) $$(x, y) = \left(\frac{a}{a^2 + b^2}, -\frac{b}{a^2 + b^2}\right).$$

By (11–3) and (11–5), the equation $(a, b) \cdot (x, y) = \mathbf{1}$ may be written in the form

$$(ax - by, ay + bx) = (1, 0).$$

Since $a^2 + b^2 \neq 0$, we may determine x and y by solving the simultaneous equations

$$\begin{cases} ax - by = 1, \\ bx + ay = 0. \end{cases}$$

The values of x and y can be found by any of the methods taken up in Section 9–1. The solution is

$$x = \frac{a}{a^2 + b^2}, \quad y = -\frac{b}{a^2 + b^2}.$$

We have, therefore, verified (11–9).

Division of (a, b) by (c, d). If $(c, d) \neq 0$, division can be defined as follows:

$$(a, b) \div (c, d) = (a, b) \cdot (u, v),$$

where (u, v) is the reciprocal of (c, d). By (11–9), we have

(11–10) $$(a, b) \div (c, d) = (a, b)\left(\frac{c}{c^2 + d^2}, -\frac{d}{c^2 + d^2}\right)$$
$$= \left(\frac{ac + bd}{c^2 + d^2}, \frac{-ad + bc}{c^2 + d^2}\right).$$

For example,
$$(5, 13) \div (3, -2) = \left(-\frac{11}{13}, \frac{49}{13}\right).$$

This result can also be verified by the method in Section 11–3.

11–2. THE STANDARD NOTATION FOR COMPLEX NUMBERS

The special number $i = (0, 1)$, defined by (11–6), has the following property:

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -(1, 0) = -\mathbf{1}. \tag{11–11}$$

We shall now show that (a, b) and the binomial form $a + bi$ are equivalent, or that

$$(a, b) = a + bi, \tag{11–12}$$

in which $a + bi$ means $a\mathbf{1} + bi$. By (11–5), (11–6), and (11–7),

$$a\mathbf{1} + bi = a(1, 0) + b(0, 1) = (a, 0) + (0, b).$$

Finally, by (11–1), we have

$$(a, 0) + (0, b) = (a, b).$$

Hence, (11–12) is established.

Real and Imaginary Parts of Complex Numbers. We call a the *real part* and b the *imaginary part* of the complex number $a + bi$. If $a = 0$ and $b \neq 0$, $a + bi$ reduces to bi, which is called a *pure imaginary number*. If $b = 0$, the complex number $a + bi$, or $a\mathbf{1} + bi$, reduces to the complex number $a\mathbf{1}$, which may be identified with the real number a. The complex numbers, then, include both the real numbers and the pure imaginary numbers as special cases.

Illustrations of various classes of numbers follow:

Some real numbers are -2, 5, and $\sqrt{3}$.
Some pure imaginary numbers are $3i$, and $-\sqrt{5}i$.
Some complex numbers are $2 + 3i$, and $\sqrt{3} - i$.

Note that the numbers -2, 5, $\sqrt{3}$, $3i$, and $-\sqrt{5}i$ may be put into the standard form $a + bi$ and written, respectively, as the complex numbers $-2 + 0i$, $5 + 0i$, $\sqrt{3} + 0i$, $0 + 3i$, and $0 - \sqrt{5}i$. Since $\mathbf{0} = 0 + 0i$, which may be written briefly as 0, we shall drop the use of bold-face $\mathbf{0}$; similarly for bold-face $\mathbf{1}$.

Conjugate Complex Numbers. The *conjugate* of a complex number $a + bi$ is defined as $a - bi$. Likewise, $a + bi$ is the conjugate of $a - bi$. Some pairs of conjugate complex numbers follow: $2i$, $-2i$; $3 + 5i$, $3 - 5i$; and $x + 2yi$, $x - 2yi$. A real number is its own conjugate.

Powers of i. It is readily seen that $i^3 = i^2 \cdot i = -i$, $i^4 = i^2 \cdot i^2 = 1$, $i^5 = i^4 \cdot i = i$, $i^6 = i^4 \cdot i^2 = -1$, and so on. Therefore, successive positive integral powers of i have only four different values, namely, $i, -1, -i$, and 1; these four values are repeated in regular order. Hence, if n is any positive integer, we have in general

$$\begin{aligned} i^{4n} &= i^4 = 1, \\ i^{4n+1} &= i, \\ i^{4n+2} &= i^2 = -1, \\ i^{4n+3} &= i^3 = -i. \end{aligned}$$

These relationships afford a simple method for evaluating powers of i, as shown by the following illustrations: $i^7 = i^{4+3} = i^3 = -i$; $i^{38} = i^{4\cdot 9+2} = i^2 = -1$; and $i^{105} = i^{4\cdot 26+1} = i$.

11–3. OPERATIONS ON COMPLEX NUMBERS IN STANDARD FORM

From the definitions given in Sections 11–1 and 11–2, it follows that $a + bi$ and $c + di$ can be added, subtracted, multiplied, and divided as if they were real binomials, except that, where i^2 appears, it is replaced by -1.

Algebraic Addition and Subtraction. Addition of two complex numbers is effected by adding their real and imaginary parts separately; and subtraction is performed by subtracting their real and imaginary parts separately. Thus, in accordance with (11–1) and (11–2),

(11–1a) $$(a + bi) + (c + di) = (a + c) + (b + d)\, i,$$

and

(11–2a) $$(a + bi) - (c + di) = (a - c) + (b - d)\, i.$$

For example,

$$(3 + 2i) + (4 - 5i) = (3 + 4) + (2 - 5)i = 7 - 3i,$$

and

$$(3 + 2i) - (4 - 5i) = (3 - 4) + (2 + 5)i = -1 + 7i.$$

We note that the sum of conjugate complex numbers is a real number, because $(a + bi) + (a - bi) = 2a$. Also, the difference of two conjugate complex numbers is a pure imaginary number, because $(a + bi) - (a - bi) = 2bi$.

Algebraic Multiplication. To find the product of two complex numbers, multiply them according to the rules of algebra, and replace i^2 by -1 in the result. Thus

$$(a + bi)(c + di) = ac + adi + bci + bdi^2.$$

Replacing i^2 by -1, we have, in agreement with (11–3),

(11–3a) $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

In many respects the notation $a + bi$ is more convenient than (a, b). In particular, the former notation makes it easier to remember how to multiply two complex numbers. For example,

$$(3 + 2i)(4 - 5i) = (12 + 10) + (-15 + 8)i = 22 - 7i.$$

The student should note that the product of two conjugate complex numbers is a non-negative real number, because $(a + bi)(a - bi) = a^2 + b^2$.

Algebraic Division. To obtain the quotient of two complex numbers, multiply the numerator and the denominator by the conjugate of the denominator. Thus, if $c + di \neq 0$,

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2i^2}$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

Therefore,

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i.$$

The right member is of the form $A + Bi$, and this equation agrees with (11–10).

Example 11–1. Reduce $\frac{1}{i}$ to the form $a + bi$.

Solution: The conjugate of i is $-i$. Then

$$\frac{1}{i} = \frac{1}{i} \cdot \frac{(-i)}{(-i)} = \frac{-i}{-i^2} = \frac{-i}{+1} = -i = 0 - i.$$

Example 11–2. Find the value of $5 + 13i$ divided by $3 - 2i$.

Solution: Represent the division as $\frac{5 + 13i}{3 - 2i}$, and multiply the numerator and the denominator by $3 + 2i$. We get

$$\frac{5 + 13i}{3 - 2i} \cdot \frac{3 + 2i}{3 + 2i} = \frac{(15 - 26) + (39 + 10)i}{9 + 4} = -\frac{11}{13} + \frac{49}{13} i.$$

EXERCISE 11–1

In each of the problems from 1 to 12, express the given quantity in the form $a + bi$ and give its conjugate. In working these problems, note that $\sqrt{-a^2} = \sqrt{a^2}\sqrt{-1} = |a| i$.

1. $\sqrt{-16}$.
2. $\sqrt{-25}$.
3. $-\sqrt{-9}$.
4. $\sqrt{-x^2}$.
5. $-\sqrt{-36a^2}$.
6. $\sqrt{-x^2y^6}$.
7. $3\sqrt{2} + 3\sqrt{-2}$.
8. $4 - 3\sqrt{-16}$.
9. $1 + 2\sqrt{-8}$.
10. $\sqrt{x^2} - \sqrt{-x^2}$.
11. $\sqrt{15} + \sqrt{-64a^3b}$.
12. $3 + \sqrt{-32a^2b^3}$.

In each of the problems from 13 to 26, compute the value of the given expression.

13. i^{11}. **14.** i^{12}. **15.** $(-i)^{13}$. **16.** $-(-i)^{17}$.
17. i^{31}. **18.** $-i^{70}$ **19.** $(-i)^{235}$. **20.** $(-i)^{602}$.
21. i^3i^{29}. **22.** $i^{37}i^{183}$. **23.** $i^{14}-(-i)^{18}$. **24.** $i^{10}+i^{20}+i^{30}$.
25. $i^{25}-i^{50}+i^{75}-i^{100}$. **26.** $i^{10}+i^{100}+i^{1000}+i^{10000}$.

In each of the problems from 27 to 36, find the values of x and y which satisfy the given equation.

27. $(2x, 3y) = (3, 1)$. **28.** $(2x, 3y) = (8, 9)$.
29. $(3x, 5y) = (18, -25)$. **30.** $(x + y), 3x + 1) = (x, 1)$.
31. $(3x + 2y, x + 5y) = (2y + 3, -19)$.
32. $x + (y - x)i = 1 + 3i$. **33.** $3x - 6 - (5 - 2y)i = 0$.
34. $2x - 4yi = 6 - 2xi$. **35.** $3x - 7 = (4 - 3y)i$.
36. $3xi - 2yi + 8x + 5y - 12 = 2x + 2yi + 5y + 6 + (x - 2)i$.

In each of the problems from 37 to 78, perform the indicated operations and reduce to the form $a + bi$.

37. $(2, -3) + (5, 6)$. **38.** $(1, -1) + (3, 2)$. **39.** $(1, \sqrt{3}) + \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.
40. $(1, -7) - (7, -3)$. **41.** $(2, 3)\cdot(1, 1)$. **42.** $(1, 1)^2$.
43. $(4, -3)\cdot(4, 3)$. **44.** $(0, -1)^3$. **45.** $(1, 0)\cdot(0, 1)\cdot(1, 1)$.
46. $(0, 1)^4$. **47.** $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^3$.
48. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)^3$. **49.** $\sqrt{-3} + \sqrt{-2} - \sqrt{9}$.
50. $\sqrt{-5} - \sqrt{-2} - 5i^2$. **51.** $(4 + 3i) + (2 - 3i)$.
52. $(-2 + 3i) - (-6 - 3i)$. **53.** $(8 + 9i) + (5 + 2i)$.
54. $(3 - 2i) - (3 + 5i)$. **55.** $(-3 + 2i) - (5 - 2i)$.
56. $(6 + 2i) + (3i + \sqrt{-1})$. **57.** $(2i + 3) + (8 - 5\sqrt{-1})$.
58. $(6 + 3i) + (3 - 5i)$. **59.** $(2 + 5i)(6 - 3i)$. **60.** $2i(5 + 3i)$.
61. $(5 + i)(6 + 2i)$. **62.** $(3 - 3i)(4i + 2)$. **63.** $(1 + i\sqrt{2})(5 + 2i)$.
64. $(5i - 8)(2i - 4)$. **65.** $(5 - 3i)(2 - 4i)$. **66.** $(3 - i) \div (2 - 5i)$.
67. $(3i + 4) \div (1 - i)$. **68.** $(2 + i)^2 \div (5 - i)$. **69.** $(2 - 3i) \div (5 - 4i)$.
70. $(3 - \sqrt{3}i) \div (4 + 5i)^2$.
71. $(3 - \sqrt{3}i)^2 + (3 + \sqrt{3}i)^2 - (4 - 3i)(i - 6)$.
72. $(6 - 2i)(1 + i)(1 - 3i) \div (4 - 5i)$.
73. $1 \div (6 - 5i)$. **74.** $(6 - 5i) \div i$. **75.** $i \div (4 - 3i)$.
76. $\dfrac{3 - 4i}{(2 + i)(3 - 2i)}$. **77.** $\dfrac{(4 + i)(i - 5)}{(6 - 5i)(2i - 7)}$. **78.** $\dfrac{(3 + 5i)(8 + 6i)}{(4 - 7i)(4 + 6i)}$.
79. Prove that complex numbers satisfy the associative and commutative laws of addition and multiplication and the distributive law.
80. Prove that if $(a + bi)(c + di) = 0$, then $a + bi = 0$ or $c + di = 0$.

11–4. GRAPHICAL REPRESENTATION

As we have seen, the complex number $a + bi$ determines a definite point P in the plane whose rectangular coordinates are $x = a$ and $y = b$. Conversely, to every point P in the plane corresponds a complex number $a + bi$ for which the values of a and b are the respective rectangular coordinates of P. See Fig. 11–1. In this system, the real numbers $a + 0i$ are represented by points on the x-axis, which is called the *axis of reals.* Pure imaginary numbers $0 + bi$ are represented by points on the y-axis, which is called the *axis of imaginaries.*

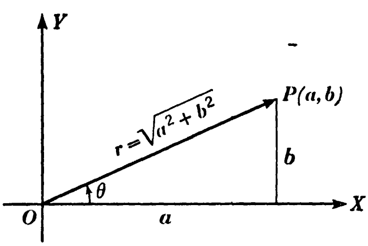

FIG. 11–1.

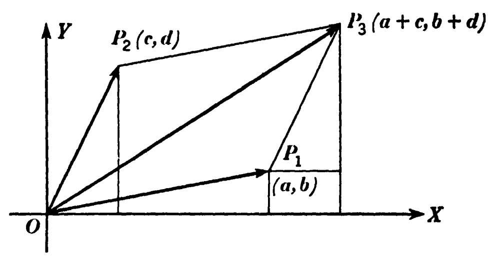

FIG. 11–2.

It is more convenient at times to represent the complex number $a + bi$ by the vector drawn from the origin to the point P. The *length* of the vector is given by the relationship $r = \sqrt{a^2 + b^2}$, and the *direction* is given by an angle θ determined from the equations $a = r \cos \theta$ and $b = r \sin \theta$.

In Fig. 11–2 is indicated the graphical addition of the two complex numbers $a + bi$ and $c + di$, which may be represented either by the points P_1 and P_2 or by the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. With the completion of the parallelogram $OP_1P_3P_2$, the sum of $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ can be represented by the diagonal $\overrightarrow{OP_3}$. Thus, either $P_3(a + c, b + d)$ or $\overrightarrow{OP_3}$ represents the sum $(a + c) + (b + d)i$. Hence, the vector which represents the sum of two complex numbers is the sum of the vectors representing the given numbers.

To subtract $c + di$ from $a + bi$ graphically, we merely add $a + bi$ and $-c -di$.

Example 11–3. Add the complex numbers $2 + 3i$ and $6 + 2i$ graphically.

Solution: Let P_1 (2, 3) represent the number $2 + 3i$ and let P_2 (6, 2) represent the number $6 + 2i$. Draw OP_1 and OP_2 in Fig. 11–3, and complete the parallelogram $OP_1P_3P_2$. Then P_3 represents the sum $8 + 5i$ of the complex numbers $2 + 3i$ and $6 + 2i$, and $\overrightarrow{OP_3}$ represents the sum of the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$.

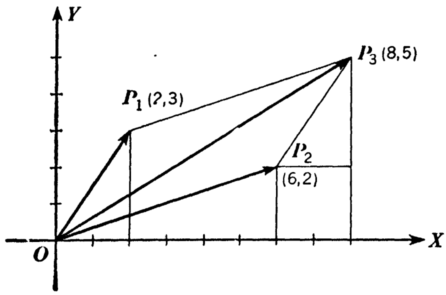

FIG. 11–3.

The difference of two complex numbers may be obtained in the same manner if we apply the relationship

$$(a + bi) - (c + di) = (a + bi) + (-c - di).$$

Let P and Q represent the numbers $a + bi$ and $c + di$, respectively, in the complex plane, as shown in Fig. 11–4. Then $-c -di$ is represented by Q', which is the reflection of Q through the origin.

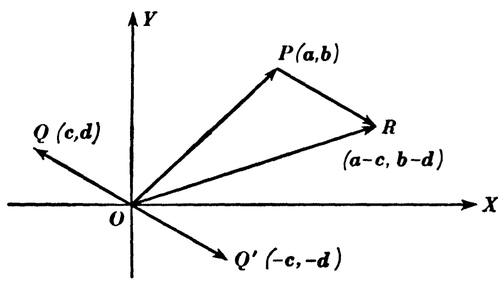

FIG. 11–4.

Let us recall how the difference of two vectors was explained in Section 6–7 and was represented graphically in Fig. 6–15. If we let the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OQ'}$ represent $a + bi$, $c + di$, and $-c -di$, respectively, then $\overrightarrow{OR}$ is the desired vector and R is the point representing the number $(a + bi) - (c + di)$.

11–5. TRIGONOMETRIC REPRESENTATION

Let the complex number $a + bi$ be represented by the radius vector drawn from the origin to the point P. Then the distance $|\overrightarrow{OP}| = r$ is called the *modulus*, or the *absolute value*, of the complex number; and the angle θ, which $\overrightarrow{OP}$ makes with the positive x-axis, is called an *argument*, *amplitude*, or *angle* of the complex number.

From Fig. 11–1, it is clear that $\frac{a}{r} = \cos\theta$ and $\frac{b}{r} = \sin\theta$, or

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta.$$

Hence,

$$a + bi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

This last expression is known as the *polar form* or the *trigonometric form* of the given complex number, as contrasted with the standard or *rectangular form* $a + bi$.

To reduce a given complex number $a + bi$ to the trigonometric form $r(\cos\theta + i\sin\theta)$, we find r and θ by means of the relationships $r = \sqrt{a^2 + b^2}$, $a = r\cos\theta$, and $b = r\sin\theta$. We have

$$a + bi = r\left(\frac{a}{r} + \frac{b}{r}i\right) = r(\cos\theta + i\sin\theta).$$

Example 11–4. Represent the complex number $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ graphically, and change the given notation to the trigonometric form.

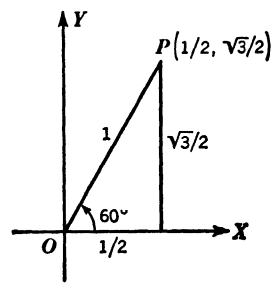

FIG. 11–5.

Solution: The point P whose rectangular coordinates are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ represents the number $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Since $a = 1/2$ and $b = \sqrt{3/2}$ are both positive, θ is a first-quadrant angle. We see from Fig. 11–5 that $r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$. The angle θ is determined from the equations $\cos\theta = 1/2$ and $\sin\theta = \sqrt{3}/2$. In this case we may let $\theta = \pi/3$ or 60°. Hence, $\frac{1}{2} + \frac{\sqrt{3}}{2}i = 1\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \cos 60° + i\sin 60°$.

Example 11–5. Express the complex number $1 - i$ in the trigonometric form.

Solution: Here $a = 1$ and $b = -1$. So $r = \sqrt{2}$, and θ is a fourth-quadrant angle determined from $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$. We thus have

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}(\cos 315° + i\sin 315°).$$

EXERCISE 11–2

In each of problems from 1 to 12, represent the complex number and its conjugate graphically.

1. $3 + 2i$.	**2.** $8 + 2i$.	**3.** $3 + 4i$.	**4.** $2 - 3i$.
5. $3 - 5i$.	**6.** $1 - i$.	**7.** i.	**8.** 1.
9. $\frac{2}{1}$.	**10.** $\frac{1}{2}(2 + \sqrt{2}i)$.	**11.** $\frac{1}{2}(1 - \sqrt{3}i)$.	**12.** $5 + 12i$.

In each of the problems from 13 to 24, perform the indicated operations graphically. Then check the result algebraically.

13. $(7 - 3i) + (-4 + i)$.
14. $(2 + 3i) - (4 - 5i)$.
15. $(3 - 6i) - (4 + 3i)$.
16. $(6 - 2i) + (6 + 2i)$.
17. $(3 + 2i) - (5 - i)$.
18. $(3 + 4i) - (-2 - 4i)$.
19. $(5 + i) + (1 - 5i)$.
20. $(3 - 2i) - (5 - 4i)$.
21. $3 - (1 - 4i) - (2 + i)$.
22. $(\sqrt{2} + i) + (1 + \sqrt{3}i) - 7i$.
23. $7 - (4 - 2i) - (-2 + i\sqrt{3})$.
24. $(4 + 3i) - (2 + 3i) - (3 + 2i)$.

In each of problems from 25 to 36, change the complex number to the trigonometric form and represent it graphically.

25. $1 + i$.
26. -5.
27. $-3i$.
28. $3 - 3\sqrt{3}i$.
29. $\frac{1}{2}(1 + \sqrt{2}i)$.
30. $\frac{1}{2}(1 - \sqrt{3}i)$.
31. $5 + 12i$.
32. $\frac{2}{i}$.
33. $\frac{6 - 4i}{2 - i} - \frac{1 - i}{3 + i}$.
34. $(3 - 4i)^2$.
35. $\frac{1}{6 + 5i}$.
36. $\frac{3 - 2i}{2 + i}$.

11–6. MULTIPLICATION AND DIVISION IN TRIGONOMETRIC FORM

Let $r_1(\cos\alpha + i\sin\alpha)$ and $r_2(\cos\beta + i\sin\beta)$ be any two complex numbers in trigonometric form. Then, their product is given by the relationship

$$\begin{aligned} r_1(\cos\alpha + i\sin\alpha)\cdot r_2(\cos\beta + i\sin\beta) &= r_1r_2[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)] \\ &= r_1r_2[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]. \end{aligned}$$

Thus, we have proved that the absolute value of the product of two complex numbers is the product of their absolute values, and an angle of the product is the sum of their angles.

The result found for the product of two complex numbers can be extended to the product of three or more complex numbers.

The quotient obtained by dividing the complex number $r_1(\cos\alpha + i\sin\alpha)$ by the complex number $r_2(\cos\beta + i\sin\beta)$ is given by the relationship

$$\begin{aligned} \frac{r_1(\cos\alpha + i\sin\alpha)}{r_2(\cos\beta + i\sin\beta)} &= \frac{r_1(\cos\alpha + i\sin\alpha)}{r_2(\cos\beta + i\sin\beta)}\cdot\frac{\cos\beta - i\sin\beta}{\cos\beta - i\sin\beta} \\ &= \frac{r_1}{r_2}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)]. \end{aligned}$$

It follows that the absolute value of the quotient of two complex numbers is the quotient of the absolute values, and an angle of the quotient is the difference of their angles.

Example 11–6. Find the product of $2(\cos 30° + i \sin 30°)$ and $3(\cos 120° + i \sin 120°)$.

Solution: By the rule for products in polar form, we have

$$\begin{aligned} &2(\cos 30° + i \sin 30°) \cdot 3(\cos 120° + i \sin 120°) \\ &\quad = 2 \cdot 3\,[\cos (30° + 120°) + i \sin (30° + 120°)] \\ &\quad = 6\,[\cos 150° + i \sin 150°] = 6\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 3\left(-\sqrt{3} + i\right). \end{aligned}$$

Example 11–7. Find the quotient when $1 - i$ is divided by $\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

Solution: From Examples 11–4 and 11–5 in Section 11–5,

$$1 - i = \sqrt{2}(\cos 315° + i \sin 315°),$$

and

$$\frac{1}{2} + \frac{\sqrt{3}}{2} i = \cos 60° + i \sin 60°.$$

Hence,

$$\begin{aligned} \frac{1 - i}{\frac{1}{2} + \frac{\sqrt{3}}{2} i} &= \sqrt{2}\,\frac{(\cos 315° + i \sin 315°)}{\cos 60° + i \sin 60°} \\ &= \sqrt{2}\,[\cos (315° - 60°) + i \sin (315° - 60°)] \\ &= \sqrt{2}\,[\cos 255° + i \sin 255°] \\ &= -\sqrt{2}\,(\cos 75° + i \sin 75°). \end{aligned}$$

From a table of trigonometric functions, we find that the result is

$$-\sqrt{2}\,[0.2588 + i\,(0.9659)].$$

Or, using exact values of cos 75° and sin 75° previously found, we obtain

$$-\sqrt{2}\left[\frac{\sqrt{2}}{4}(\sqrt{3} - 1) + i\,\frac{\sqrt{2}}{4}(\sqrt{3} + 1)\right] = -\frac{1}{2}\,[(\sqrt{3} - 1) + i\,(\sqrt{3} + 1)].$$

11–7. De MOIVRE'S THEOREM

If we extend the law of multiplication of the preceding section to n factors, we have

$$\begin{aligned} &[r_1(\cos \theta_1 + i \sin \theta_1)]\,[r_2(\cos \theta_2 + i \sin \theta_2)] \cdots [r_n(\cos \theta_n + i \sin \theta_n)] \\ &\qquad = r_1 r_2 \cdots r_n[\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i \sin (\theta_1 + \theta_2 + \cdots + \theta_n)]. \end{aligned}$$

If now we put $r_1 = r_2 = \cdots = r_n = r$ and $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$, it follows that

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).$$

This result is known as *De Moivre's Theorem.*

Although we have derived De Moivre's Theorem only for integral values of n, it can be shown to hold for all real values of n, if properly interpreted.

Example 11–8. Find the value of $(1 - i)^4$ by De Moivre's Theorem.

Solution: Since $1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$, the polar form of $1 - i$ is $\sqrt{2}\,(\cos 315° + i \sin 315°)$. Hence, by De Moivre's Theorem,

$$\begin{aligned}(1 - i)^4 &= [\sqrt{2}\,(\cos 315° + i \sin 315°)]^4\\ &= (\sqrt{2})^4\,[\cos (4 \cdot 315°) + i \sin (4 \cdot 315°)]\\ &= 4(\cos 1260° + i \sin 1260°)\\ &= 4(\cos 180° + i \sin 180°) = -4.\end{aligned}$$

Example 11–9. Derive formulas for $\cos 2\theta$ and $\sin 2\theta$ by De Moivre's Theorem.

Solution: By De Moivre's Theorem for $n = 2$, we have

$$\cos 2\theta + i \sin 2\theta = (\cos \theta + i \sin \theta)^2 = \cos^2 \theta + (2 \cos \theta \sin \theta)\, i - \sin^2 \theta.$$

The two sides are equal only if the corresponding real and imaginary parts are equal. Hence,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta,$$

and

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

11–8. ROOTS OF COMPLEX NUMBERS

Let $\rho(\cos \phi + i \sin \phi)$ be an nth root of the complex number $r(\cos \theta + i \sin \theta)$, where $0° \leqq \theta \leqq 360°$. Then

$$[\rho(\cos \phi + i \sin \phi)]^n = r(\cos \theta + i \sin \theta).$$

By De Moivre's theorem, this leads to

$$(11\text{–}13) \qquad \rho^n(\cos n\phi + i \sin n\phi) = r(\cos \theta + i \sin \theta).$$

Our problem now is to find all non-negative numbers ρ and all angles ϕ for which (11–13) is satisfied. Separating real and imaginary parts, we have

$$(11\text{–}14) \qquad \rho^n \cos n\phi = r \cos \theta, \quad \rho^n \sin n\phi = r \sin \theta.$$

Squaring and adding, we obtain

$$\rho^{2n}(\cos^2 n\phi + \sin^2 n\phi) = r^2(\cos^2 \theta + \sin^2 \theta).$$

Therefore, $\rho^{2n} = r^2$, since $\cos^2 \alpha + \sin^2 \alpha = 1$. The absolute value ρ is then given by the equation

$$(11\text{–}15) \qquad \rho = \sqrt[n]{r}.$$

From (11–14) we then have

$$\cos n\phi = \cos \theta, \quad \sin n\phi = \sin \theta.$$

It is clear from these equations that the angles $n\phi$ and θ can differ only by a multiple of 2π or $360°$. More precisely,

$$(11\text{–}16) \qquad n\phi = \theta + k \cdot 360°, \quad \text{or} \quad \phi = \frac{\theta}{n} + \frac{k \cdot 360°}{n},$$

where k is any integer.

For $k = 0, 1, 2, \cdots, (n-1)$ in (11–16), we obtain n distinct values of the angle, all of which are non-negative and less than 2π or $360°$. Corresponding to these angles we obtain n distinct roots given by the formula

$$(11\text{–}17) \qquad \sqrt[n]{r}\left[\cos\frac{\theta + k\cdot 360°}{n} + i\sin\frac{\theta + k\cdot 360°}{n}\right].$$

For example, for $k = 0$, we obtain one nth root, called the *principal root,* with absolute value $r^{1/n}$ and angle $\frac{\theta}{n}$; for $k = 1$, we have a second root, with absolute value $r^{1/n}$ and angle $\frac{\theta + 360°}{n}$; and so on to $k = n - 1$. The value $k = n$ would yield the same root as $k = 0$, since $\frac{\theta + n\cdot 360°}{n} = \frac{\theta}{n} + 360°$, and $\cos\left(\frac{\theta}{n} + 360°\right) = \cos\frac{\theta}{n}$ and $\sin\left(\frac{\theta}{n} + 360°\right) = \sin\frac{\theta}{n}$. Similarly, $k = n + 1$ yields the same root as $k = 1$, and so on. This means that only n distinct roots exist.

It is interesting to note that the points which represent the roots are equally spaced on a circle whose radius is $\sqrt[n]{r}$ and whose center is the origin. This fact is illustrated in the following example and Fig. 11–6.

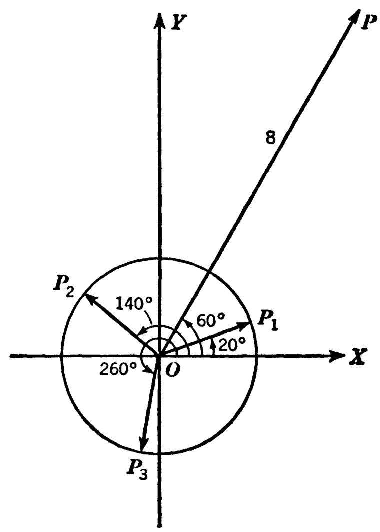

FIG. 11–6.

Example 11–10. Find the three cube roots of $8(\cos 60° + i\sin 60°)$.

Solution: The three cube roots are found by evaluating (11–17). Thus, we have

$$\begin{aligned}\sqrt[3]{8(\cos 60° + i\sin 60°)} &= \sqrt[3]{8}\left(\cos\frac{60° + k\cdot 360°}{3} + i\sin\frac{60° + k\cdot 360°}{3}\right)\\ &= 2[\cos(20° + k\cdot 120°) + i\sin(20° + k\cdot 120°)].\end{aligned}$$

As just shown, the substitution of $k = 0, 1, 2$ yields the three required roots. Hence, for $k = 0$, we have $2(\cos 20° + i\sin 20°)$; for $k = 1$, the root is $2(\cos 140° + i\sin 140°)$; and for $k = 2$, the root is $2(\cos 260° + i\sin 260°)$. The roots are represented by the equally spaced vectors $\overrightarrow{OP}_1$, $\overrightarrow{OP}_2$, and $\overrightarrow{OP}_3$, terminating on the circle whose radius is 2 and making angles of 20°, 140°, and 260°, respectively, with the positive x-axis.

EXERCISE 11–3

In each of the problems from 1 to 18, perform the indicated operations by first expressing the complex number in polar form. Express the answer in rectangular form.

1. $(1 + i)(1 - \sqrt{3}i)$.
2. $(-1 + i)(\sqrt{3} + i)$.
3. $(-1 + \sqrt{3}i)(\sqrt{3} + i)$.
4. $\dfrac{1 - i}{1 + \sqrt{3}i}$.
5. $\dfrac{i - \sqrt{3}}{1 + i}$.
6. $\dfrac{2 + 2i}{\sqrt{3} + 3i}$.
7. $\dfrac{(1 - i)(-1 + \sqrt{3}i)}{\sqrt{3} + i}$.
8. $\dfrac{(-1 + i)(\sqrt{3} - 3i)}{1 + i}$.
9. $\left(\dfrac{1 - \sqrt{3}i}{2}\right)^3$.
10. $(-1 + \sqrt{3}i)^2$.
11. $\left(\dfrac{1 - \sqrt{3}i}{2}\right)^8$.
12. $\dfrac{1 + i}{(2 + i)(3 + i)}$.
13. $\dfrac{i - 1}{(1 + i)(3 + 4i)}$.
14. $\dfrac{1 - \sqrt{3}i}{(2 + 3i)(\sqrt{3} + i)}$.
15. $\dfrac{(2 + 3i)(1 - i)}{2 - 2i}$.
16. $\dfrac{3 + 2i}{3 - 2i} \div (1 + 2i)$.
17. $(\sqrt{2} - \sqrt{2}i)^{10}$.
18. $\dfrac{(2 - 5i)^2(1 + 3i)}{(3i)^3}$.
19. $\left[\frac{1}{2}(1 - \sqrt{2}i)\right]^{100}$.
20. $\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i\right)^{70}$.
21. $\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^{150}$.

In each of the problems from 22 to 29, find roots as directed and represent them graphically.

22. Find two distinct square roots of $9(\cos 50° + i \sin 50°)$.
23. Find four distinct fourth roots of $16(\cos 36° + i \sin 36°)$.
24. Find three distinct cube roots of $27(\cos 165° + i \sin 165°)$.
25. Find five distinct fifth roots of -32.
26. Find the three cube roots of 1.
27. Find the four fourth roots of 1.
28. Find the two square roots of i.
29. Find the three cube roots of $-\dfrac{1}{2}(1 + \sqrt{3}i)$.

In each of the problems from 30 to 34, the complex numbers E, I, and Z designate voltage, current, and impedance, respectively, and $E = IZ$.

30. Compute E when $I = 5 + 4i$ amperes and $Z = 30 - 8i$ ohms.

31. Compute I when $E = 110 + 30i$ volts and $Z = 20 - 15i$ ohms.

32. Compute Z when $I = 4 + 3i$ amperes and $E = 115$ volts.

33. When two impedances Z_1 and Z_2 are connected in parallel, the equation $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$ determines an equivalent impedance Z. Compute Z when $Z_1 = 5 + 4i$ ohms and $Z_2 = 8 - 6i$ ohms.

34. If z and $\bar{z}$ are conjugate complex numbers, prove that

$$|z + \bar{z}|^2 + |z - \bar{z}|^2 = 4|z|^2.$$

12 Equations in Quadratic Form

12–1. QUADRATIC EQUATIONS IN ONE UNKNOWN

This chapter provides an extension of the work on linear equations to *second-degree,* or *quadratic, equations.* Consider a quadratic equation in one unknown written in the form

$$ax^2 + bx + c = 0 \qquad (a \neq 0), \tag{12–1}$$

where a, b, and c are given real numbers. This equation is called the *general quadratic equation* in x, and is said to be in *standard form.*

If $b \neq 0$, (12–1) is called a *complete quadratic equation;* if $b = 0$, it is called a *pure quadratic equation.* Thus, $3x^2 - x + 4 = 0$ is a complete quadratic equation in which $a = 3$, $b = -1$, and $c = 4$; and $x^2 - 2 = 0$ is a pure quadratic with $a = 1$ and $c = -2$.

In Section 12–4 we shall prove that every quadratic equation has two and only two solutions or roots. The roots may be equal or unequal, and they may be real or complex. Their natures depend on the values of a, b, and c. We shall consider the methods in general use for finding these roots, and we shall then apply them as well to the solution of equations which are not quadratic in x but which can be written as quadratic equations in expressions involving the unknown.

12–2. SOLUTION OF QUADRATIC EQUATIONS BY FACTORING

If the left side of a quadratic equation in standard form can be factored, the solution of the equation depends on the following important principle:

The product of two or more numbers equals zero if and only if at least one of the factors is equal to zero.

That is,

$$A \cdot B = 0 \text{ if and only if } A = 0 \text{ or } B = 0.$$

In practice, we apply this principle by equating to zero each linear factor of the left side of the given quadratic equation, and solving the resulting linear equations. The following examples will illustrate its application.

Example 12–1. Solve $2x^2 - 7x + 6 = 0$ by factoring.

Solution: To find the values of x which satisfy the equation $2x^2 - 7x + 6 = 0$, write the left side in the factored form

$$(x - 2)(2x - 3) = 0.$$

This product equals zero if and only if either

$$x - 2 = 0 \qquad \text{or} \qquad 2x - 3 = 0.$$

Hence, $x = 2$ or $x = 3/2$. Moreover, 2 and 3/2 are solutions, because both 2 and 3/2 satisfy $2x^2 - 7x + 6 = 0$. Thus,

$$2(2)^2 - 7(2) + 6 = 8 - 14 + 6 = 0,$$

and

$$2(3/2)^2 - 7(3/2) + 6 = 9/2 - 21/2 + 6 = 0.$$

Example 12–2. *a*) Solve the equation $2 \sin^2 x - \sin x - 1 = 0$ for $\sin x$. *b*) Find all non-negative angles x less than 360° which satisfy this equation.

Solution: *a*) Factor the given equation to obtain

$$(\sin x - 1)(2 \sin x + 1) = 0.$$

Since $\sin x - 1 = 0$ or $2 \sin x + 1 = 0$, it follows that $\sin x = 1$ or $\sin x = -1/2$.
Check:

$$2(1)^2 - (1) - 1 = 2 - 1 - 1 = 0,$$

and

$$2\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = \frac{2}{4} + \frac{1}{2} - 1 = 1 - 1 = 0.$$

b) When $\sin x = 1$, $x = 90°$; when $\sin x = -1/2$, $x = 210°$ or $330°$. Therefore, $x = 90°$ or $210°$ or $330°$.

Check: $2 \sin^2 90° - \sin 90° - 1 = 2 - 1 - 1 = 0$,

$$2 \sin^2 210° - \sin 210° - 1 = 2\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = 0,$$

and

$$2 \sin^2 330° - \sin 330° - 1 = 2\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = 0.$$

Note that this equation is a quadratic in which the unknown is a trigonometric function of the angle x. We thus have only two values of $\sin x$ which satisfy the equation. The determination of the angle, however, goes beyond the algebraic solution of the quadratic, and it may happen that there are more than two values of x which satisfy the equation. For this reason, it is recommended that all solutions be checked by substituting in the original equation.

Example 12–3. Solve the equation $3 \sec x = 2 \cos x - 1$ for all non-negative values of x less than 2π radians.

Solution: Since $\sec x = \dfrac{1}{\cos x}$, we can write $\dfrac{3}{\cos x} = 2 \cos x - 1$. We then clear of fractions, transpose, and factor, to obtain

$$2 \cos^2 x - \cos x - 3 = (2 \cos x - 3)(\cos x + 1) = 0.$$

Hence, $\cos x = 3/2$ or $\cos x = -1$. When $\cos x = -1$, $x = \pi$. There is no real number x for which $\cos x = 3/2$.

Check:

$$3 \sec \pi = 2 \cos \pi - 1, \quad \text{or} \quad 3(-1) = 2(-1) - 1.$$

It should be noted that factoring provides a method of solving any pure quadratic equation $ax^2 + c = 0$. Thus, the equation is equivalent to

$$x^2 + \frac{c}{a} = \left(x + \sqrt{\frac{-c}{a}}\right)\left(x - \sqrt{\frac{-c}{a}}\right) = 0,$$

which gives

$$x + \sqrt{\frac{-c}{a}} = 0 \quad \text{or} \quad x - \sqrt{\frac{-c}{a}} = 0,$$

or $x = \pm\sqrt{\dfrac{-c}{a}}$. This result agrees with that given by writing $x^2 = \dfrac{-c}{a}$ and then simply extracting square roots of both sides to obtain $x = \pm\sqrt{\dfrac{-c}{a}}$. Note that the roots are real when $\dfrac{c}{a} \leqq 0$ and pure imaginary when $\dfrac{c}{a} > 0$.

EXERCISE 12–1

Solve each of the following equations for x or θ. In each of the problems from 9 to 20, find all non-negative angles θ less than 360° which satisfy the given equation. Check all solutions.

1. $x^2 + 7x = 0$.

2. $x^2 - 10x + 21 = 0$.

3. $2x(4x + 5) = -3$.

4. $x^2 + 6x - 27 = 0$.

5. $x^2 - x = 6$.

6. $6x^2 - 4x - 192 = 0$.

7. $x^3 + 27 - 3x(x + 3) = 0$.

8. $16x^2 - a^2 + 2ab - b^2 = 0$.

9. $\sin^2 \theta - \sin \theta = 0$.

10. $\sin \theta = \csc \theta$.

11. $\sin \theta + 1 = 2 \csc \theta$.

12. $2 \tan^2 \theta + 3 \tan \theta - 2 = 0$.

13. $\sec \theta (\sec \theta + 6) = 16$.

14. $3 \cos^2 \theta - 8 \sin \theta = 0$.

15. $(\cot \theta - 1)(\cot \theta + 2) = 4$.

16. $\dfrac{2 \csc \theta}{3} - \dfrac{1}{6} = \dfrac{12}{\csc \theta} - \dfrac{4}{2 \csc \theta}$.

17. $\dfrac{\cot \theta + 4}{\cot \theta + 6} = \dfrac{143}{(\cot \theta + 6)^2}$.

18. $\dfrac{8}{3 \cos \theta + 3} + \dfrac{2 + 3 \cos \theta}{3 \cos \theta + 4} + 1 = 0$.

19. $3 \sin^2 \theta = 4 \sin^3 \theta - 10 \sin \theta$.

20. $\dfrac{3}{4 \cos^2 \theta} - \dfrac{7}{8 \cos \theta} - \dfrac{5}{2} = 0$.

12–3. COMPLETING THE SQUARE

The method developed here is based on the fact that we can make any binomial of the form $x^2 + kx$ into a perfect square if we add to

it the square of one-half the coefficient of x. To make this clear, let us recall from Section 1–18 the formula for a perfect-square trinomial. The formula is

$$(x + a)^2 = x^2 + 2ax + a^2.$$

Since the coefficient of x in $x^2 + kx$ is k, the square of one-half of the coefficient is $\left(\frac{k}{2}\right)^2$ or $\frac{k^2}{4}\cdot$ Adding this to $x^2 + kx$, we have

$$x^2 + kx + \frac{k^2}{4} = \left(x + \frac{k}{2}\right)^2\cdot$$

Thus, the left member is a perfect square, namely, the square of $x + \frac{k}{2}\cdot$

Applicability of the procedure to a variety of processes, including solution of quadratic equations, is illustrated in the following examples.

Example 12–4. Solve $x^2 - 2x - 4 = 0$ by completing the square.

Solution: We first transpose the constant term, so that the left side will be of the form $x^2 + kx$. Hence, the equation becomes

$$x^2 - 2x \qquad = 4.$$

Now the quantity $(-1)^2 = 1$ is added to the left side to make it a perfect square. To obtain an equivalent equation, the same quantity is added to the right side also. The result is

$$x^2 - 2x + 1 = 5,$$

or

$$(x - 1)^2 = 5.$$

Taking square roots of both sides, we have

$$x - 1 = \pm\sqrt{5}.$$

So the desired solutions are $x = 1 + \sqrt{5}$ and $x = 1 - \sqrt{5}$.

Check:

$$(1 + \sqrt{5})^2 - 2(1 + \sqrt{5}) - 4 = 1 + 2\sqrt{5} + 5 - 2 - 2\sqrt{5} - 4 = 0,$$

and

$$(1 - \sqrt{5})^2 - 2(1 - \sqrt{5}) - 4 = 1 - 2\sqrt{5} + 5 - 2 + 2\sqrt{5} - 4 = 0.$$

Example 12–5. Solve $2x^2 - 5x + 3 = 0$ by completing the square.

Solution: Transpose the constant term to obtain

$$2x^2 - 5x \qquad = -3.$$

Since the coefficient of x^2 is not 1, we make it 1 by dividing both sides by 2. Then we have

$$x^2 - \frac{5}{2}x = -\frac{3}{2}\cdot$$

The square of half the coefficient of x is $\left(\frac{1}{2}\left(-\frac{5}{2}\right)\right)^2 = \frac{25}{16}\cdot$ Add this number to both sides, thus making the left side a perfect square. The result is

$$x^2 - \frac{5}{2}x + \frac{25}{16} = -\frac{3}{2} + \frac{25}{16} = \frac{1}{16}\cdot$$

or

$$\left(x - \frac{5}{4}\right)^2 = \frac{1}{16}\cdot$$

When we take square roots of both sides, we have $x - \frac{5}{4} = \pm\frac{1}{4}\cdot$ Solving for x, we obtain

$$x = \frac{5}{4} \pm \frac{1}{4}\cdot$$

That is,

$$x = \frac{3}{2} \quad \text{or} \quad x = 1.$$

Check:

$$2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3 = \frac{9}{2} - \frac{15}{2} + 3 = 0,$$

and

$$2(1)^2 - 5(1) + 3 = 2 - 5 + 3 = 0.$$

Example 12–6. Reduce $x^2 + y^2 - 4x + 6y + 4 = 0$ to the form $(x - h)^2 + (y - k)^2 = r^2$.

Solution: The solution of this problem requires that we complete the square of the terms containing y as well as the square of those containing x. Hence, for convenience, we write the equation in the form

$$(x^2 - 4x \qquad) + (y^2 + 6y \qquad) = -4.$$

When we complete the squares in the parentheses, the equation becomes

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) = -4 + 4 + 9.$$

Thus, the solution is

$$(x - 2)^2 + (y + 3)^2 = 9.$$

Example 12–7. Reduce $9x^2 - 4y^2 - 18x - 16y - 43 = 0$ to the form

$$A(x - h)^2 - B(y - k)^2 = C.$$

Solution: Write the equation in the form $9(x^2 - 2x \qquad) - 4(y^2 + 4y \qquad) = 43.$ Complete the squares in the parentheses to obtain

$$9(x^2 - 2x + 1) - 4(y^2 + 4y + 4) = 43 + 9 - 16.$$

Note that the numbers 1 and 4, which are added within the parentheses to complete the squares, must be multiplied by the coefficients 9 and -4, respectively, to determine the numbers that are added to the right side.

The reduced form is, therefore,

$$9(x - 1)^2 - 4(y + 2)^2 = 36.$$

Example 12–8. Reduce $\sqrt{3x^2 + 4x - 4}$ to the form $\sqrt{a((x - h)^2 - k^2)}$.

Solution: For convenience, work with the quantity $3x^2 + 4x - 4$ without the radical sign until the final result is obtained. Hence, write

$$3x^2 + 4x - 4 = 3\left(x^2 + \frac{4}{3}x - \frac{4}{3}\right)\cdot$$

Complete the square of the terms in x and simplify to obtain

$$3\left(x^2+\frac{4}{3}x-\frac{4}{3}\right)=3\left(\left(x^2+\frac{4}{3}x+\frac{4}{9}\right)-\frac{4}{9}-\frac{4}{3}\right)=3\left(\left(x+\frac{2}{3}\right)^2-\frac{16}{9}\right).$$

Now, write this result under the radical sign to obtain $\sqrt{3\left(\left(x+\frac{2}{3}\right)^2-\frac{16}{9}\right)}$.

Comparing this with the required form $\sqrt{a((x-h)^2-k^2)}$, we see that we may choose $a=3$, $h=-2/3$, and $k=\pm 4/3$.

EXERCISE 12–2

In each of problems from 1 to 15, solve the given equation by completing the square. In each of the problems from 9 to 15, find all non-negative angles θ less than 360° which satisfy the given equation. Check all solutions.

1. $x^2-8x=20$. **2.** $x^2+10x=40$. **3.** $x^2-7x=30$.
4. $x^2+x+1=0$. **5.** $x^2+x+2=0$. **6.** $6x^2-5x-1=0$.
7. $x^2+x-5=0$. **8.** $2x^2=3x+9$. **9.** $\tan^2\theta=2\tan\theta+1$.
10. $\cos^2\theta=\dfrac{2}{\sec\theta}+3$. **11.** $\dfrac{1}{\sec\theta+2}=\dfrac{\sec\theta}{3}$.
12. $1+\tan^2\theta=\sec\theta+3$. **13.** $\dfrac{\csc\theta+2}{\csc^2\theta-1}=2$.
14. $3\cot^2\theta+\cot\theta=3-4\cot\theta$. **15.** $\sec\theta-\cos\theta=2$.

In each of the problems from 16 to 25, reduce the equation to the form $A(x-h)^2+B(y-k)^2=C$.

16. $x^2-4y^2-2x+1=0$. **17.** $x^2+4y^2-6x+16y+21=0$.
18. $4x^2+9y^2+32x-18y+37=0$. **19.** $x^2+4y^2-10x-40y+109=0$.
20. $9x^2+4y^2-8y-32=0$. **21.** $4x^2+9y^2-16x-18y-11=0$.
22. $x^2-9y^2-4x+36y-41=0$. **23.** $4x^2-9y^2+32x+36y+64=0$.
24. $5y^2-4x^2+50y+32x+41=0$. **25.** $3x^2-y^2+20x-2y+11=0$.

In each of the following problems, express the quantity inside the radical or parentheses in the form $a((x-h)^2\pm k^2)$.

26. $\sqrt{x^2-6x-40}$. **27.** $\sqrt{2x^2-16x+41}$. **28.** $(4y^2-20y-76)^{3/2}$.
29. $\dfrac{1}{\sqrt{x^2-6x-7}}$. **30.** $(x^2-6x+34)^{2/3}$. **31.** $(2x^2+28x+34)^{-1/2}$.
32. $\sqrt{(9x^2+48x+23)^3}$. **33.** $(9x^2+24x+25)^{-1/3}$. **34.** $(7x^2-14x+11)^{-3}$.

12–4. SOLUTION OF QUADRATIC EQUATIONS BY THE QUADRATIC FORMULA

By applying the method of completing the square to the general quadratic equation (12–1), we can obtain a formula for the roots, either real or complex, of any quadratic equation whatever. The general equation is

(12–1) $$ax^2+bx+c=0.$$

Transpose the constant term c, and obtain

$$ax^2+bx\quad=-c.$$

Dividing by a, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$ to both sides to obtain

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a},$$

which becomes

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Extracting square roots of both sides gives

$$x + \frac{b}{2a} = \frac{\pm \sqrt{b^2 - 4ac}}{2a}.$$

Solving for x, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, to solve a quadratic equation, put it into the standard form $ax^2 + bx + c = 0$, and substitute the coefficients a, b, and c in the formula just derived to obtain the roots

$$(12\text{–}2)\quad x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

That these numbers x_1 and x_2 are solutions of the given quadratic equation is shown by substituting each of them in (12–1). The details of this substitution for x_1 follow:

$$\begin{aligned} ax_1^2 + bx_1 + c &= a\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^2 + b\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + c \\ &= a\left(\frac{b^2 - 2b\sqrt{b^2 - 4ac} + (b^2 - 4ac)}{4a^2}\right) \\ &\quad + \frac{-b^2 + b\sqrt{b^2 - 4ac}}{2a} + c \\ &= \frac{b^2 - 2ac - b\sqrt{b^2 - 4ac}}{2a} + \frac{-b^2 + b\sqrt{b^2 - 4ac}}{2a} + c \\ &= -\frac{2ac}{2a} + c = -c + c = 0. \end{aligned}$$

Hence, the number x_1 satisfies the equation $ax^2 + bx + c = 0$. A similar computation shows that x_2 is also a solution of $ax^2 + bx + c = 0$. Consequently, the two expressions given for x in (12–2) are actually roots of (12–1). In Section 12–7 we shall study the expressions in (12–2) further and shall determine when the roots are distinct and when they are real.

Example 12–9. Solve $5x^2 - 6x - 8 = 0$ by the quadratic formula.

Solution: Here $a = 5$, $b = -6$, $c = -8$. Substituting these values in the formula, we obtain

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - (4)(5)(-8)}}{2 \cdot 5}$$

$$= \frac{6 \pm \sqrt{36 + 160}}{10} = \frac{6 \pm \sqrt{196}}{10} = \frac{6 \pm 14}{10}.$$

Therefore, $x_1 = \frac{6 + 14}{10} = 2$ and $x_2 = \frac{6 - 14}{10} = -\frac{4}{5}.$

These values are seen to satisfy the original equation when substituted for x.

Example 12–10. Solve $x^2 - x + 2 = 0$ by the formula.

Solution: Since $a = 1$, $b = -1$, $c = 2$, we have

$$x = \frac{1 \pm \sqrt{1 - 8}}{2} = \frac{1 \pm \sqrt{-7}}{2} = \frac{1 \pm \sqrt{7}\,i}{2}.$$

Hence,

$$x_1 = \frac{1 + \sqrt{7}\,i}{2} \quad \text{and} \quad x_2 = \frac{1 - \sqrt{7}\,i}{2}.$$

Check:

$$\left(\frac{1 + \sqrt{7}\,i}{2}\right)^2 - \frac{1 + \sqrt{7}\,i}{2} + 2 = \frac{-6 + 2\sqrt{7}\,i}{4} - \frac{1 + \sqrt{7}\,i}{2} + 2$$

$$= \frac{-6 + 2\sqrt{7}\,i - 2 - 2\sqrt{7}\,i + 8}{4} = 0.$$

Similarly, the second root may be checked.

Example 12–11. Solve $2 \sin^2 x + 3 \cos x - 3 = 0$ by the formula, determining all non-negative angles x less than 360°.

Solution: We make use of the identity $\sin^2 x + \cos^2 x = 1$ to transform the given equation into one involving a single trigonometric function of x.

Replacing $\sin^2 x$ by $1 - \cos^2 x$, we have

$$2(1 - \cos^2 x) + 3 \cos x - 3 = 0.$$

Simplifying, we obtain

$$2 \cos^2 x - 3 \cos x + 1 = 0.$$

Solving for $\cos x$ by the formula, we find

$$\cos x = \frac{3 \pm \sqrt{9 - 8}}{4} = \frac{3 \pm 1}{4}.$$

Hence, $\cos x = 1$ or $1/2$. Therefore, $x = 0°$ or $60°$ or $300°$.

The solutions may be checked by substitution in the original equation.

Example 12–12. Solve $\cos x \tan x + \sin^2 x = 1 - \sin x$ by the formula, determining all non-negative values of x less than 360°.

Solution: Making use of the identity $\tan x = \frac{\sin x}{\cos x}$, we have

$$\cos x \cdot \frac{\sin x}{\cos x} + \sin^2 x = 1 - \sin x.$$

Transposing and simplifying, we get

$$\sin^2 x + 2 \sin x - 1 = 0.$$

Solving for $\sin x$ by the quadratic formula, we obtain

$$\sin x = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}.$$

The value $\sin x = -1 + \sqrt{2}$ is one solution. However, $\sin x = -1 - \sqrt{2}$ must be excluded, since the sine of an angle cannot be numerically greater than 1.

Check: For $\sin x = \sqrt{2} - 1$, we find that

$$\cos x = \sqrt{2\sqrt{2} - 2} \quad \text{and} \quad \tan x = \frac{\sqrt{2} - 1}{\sqrt{2\sqrt{2} - 2}}.$$

Substituting these values in the original equation, we have

$$\sqrt{2\sqrt{2} - 2} \cdot \frac{\sqrt{2} - 1}{\sqrt{2\sqrt{2} - 2}} + (\sqrt{2} - 1)^2 = 1 - (\sqrt{2} - 1).$$

This reduces to

$$\sqrt{2} - 1 + 3 - 2\sqrt{2} = 1 - \sqrt{2} + 1$$

or

$$2 - \sqrt{2} = 2 - \sqrt{2}.$$

From the table of trigonometric functions, we have $x = 24°28'$ or $155°32'$.

EXERCISE 12–3

Solve each of the following equations for x or θ by the quadratic formula. In each of the problems from 14 to 24, find all non-negative angles θ less than 360° which satisfy the equation. Check all solutions.

1. $x^2 + 6x - 7 = 0.$

2. $x^2 - x - 20 = 0.$

3. $x^2 + 2x - 5 = 0.$

4. $x^2 + x + 1 = 0.$

5. $x^2 + 2x + 1 = 0.$

6. $7x^2 - 8x - 9 = 0.$

7. $2x^2 + 3x + 2 = 0.$

8. $9x^2 - 7x - 5 = 0.$

9. $(2x - 1)^2 - 2(2x - 1) - 8 = 0.$

10. $(a^2 - b^2)x^2 - 4abx - (a^2 - b^2) = 0.$

11. $\dfrac{5}{x + 2} - \dfrac{1}{2x + 3} = 4.$

12. $\dfrac{5}{x - 3} + \dfrac{8}{1 - x} = \dfrac{3}{x + 1}.$

13. $\dfrac{1}{x} + \dfrac{1}{x + 3} + \dfrac{1}{3x + 5} = 0.$

14. $\cot^2 \theta + 3 = \dfrac{4}{\tan \theta}.$

15. $\dfrac{5}{\tan \theta} - \dfrac{2}{\cot \theta} = 3.$

16. $\csc \theta - 3 = \sin \theta.$

17. $\sin^2 \theta - 2 \sin \theta = \sin \theta + 3.$

18. $16 \sec^2 \theta - 8 \sec \theta + 1 = 0.$

19. $\dfrac{3}{2} + \dfrac{2}{\cos \theta} + \dfrac{3}{2 \cos^2 \theta} = 0.$

20. $\dfrac{\cot \theta + 2}{2 \cot \theta - 3} - \dfrac{2 \cot \theta - 1}{\cot \theta} = 0.$

21. $(\sec \theta + 1)(\sec \theta + 2) = \sec \theta + 3.$

22. $(\sin \theta + 3)(3 - \sin \theta) = 3(\sin \theta + 3).$

23. $(\csc \theta + 2)^2 + \csc \theta = 1.$

24. $\dfrac{\cos^2 \theta}{\cos \theta - 1} + \dfrac{1}{2} = 0.$

12–5. EQUATIONS INVOLVING RADICALS

Sometimes an equation in which the unknown appears under a radical sign can be reduced to a quadratic by raising both sides to

a power sufficient to remove the radical. The process must be repeated until the unknown no longer occurs under a radical.

The operation of raising both sides of an equation to a power may lead to an equation redundant with respect to the original; that is, the final equation may possess roots that are not roots of the original equation. Such roots are called *extraneous roots*. For this reason, every root obtained must be checked by substitution.

Example 12–13. Solve the equation $\sqrt[3]{x^2 - 3x + 4} = 2$.

Solution: Cube both sides to obtain $x^2 - 3x + 4 = 8$. Transpose, and get

$$x^2 - 3x - 4 = 0.$$

Factoring and solving for x, we find that

$$x = -1 \quad \text{or} \quad x = 4.$$

Check:

$$\sqrt[3]{(-1)^2 - 3(-1) + 4} = \sqrt[3]{1 + 3 + 4} = \sqrt[3]{8} = 2,$$

and

$$\sqrt[3]{(4)^2 - 3(4) + 4} = \sqrt[3]{8} = 2.$$

Hence, both -1 and 4 are roots.

Example 12–14. Solve the equation $\sqrt{2x - 1} - \sqrt{x + 3} = 1$.

Solution: Transpose one radical to obtain

$$\sqrt{2x - 1} = 1 + \sqrt{x + 3}.$$

When both sides are squared, the result is

$$2x - 1 = 1 + 2\sqrt{x + 3} + x + 3.$$

Combining like terms, we obtain

$$x - 5 = 2\sqrt{x + 3}.$$

Now we square both sides to get

$$x^2 - 10x + 25 = 4(x + 3).$$

Transposing and combining gives

$$x^2 - 14x + 13 = 0.$$

By factoring and solving, we find that

$$x = 1 \quad \text{or} \quad x = 13.$$

Check:

$$\sqrt{2(1) - 1} - \sqrt{1 + 3} = 1 - 2 \neq 1,$$

and

$$\sqrt{2(13) - 1} - \sqrt{13 + 3} = 5 - 4 = 1.$$

Hence, 13 is a root, but 1 is not.

EXERCISE 12–4

Solve each of the following equations. In each case check for extraneous roots

1. $\sqrt{x - 2} = 4$. **2.** $\sqrt{3x + 4} = 2$.

3. $\sqrt{x + 5} = 1$. **4.** $\sqrt[3]{3x - 1} = 7$.

5. $\sqrt[4]{x^2 - 16} = 2x^{1/4}$. **6.** $\sqrt{x^2 - 2} = \sqrt{2x + 6}$.

7. $x - 3 - \sqrt{x - 1} = 0$.

8. $x - 5x^{1/2} + 6 = 0$.

9. $\sqrt{2x + 3} = 4 - 3x$.

10. $\sqrt{x - 1} + \sqrt{x - 3} = 2$.

11. $\sqrt{5 - x} + \sqrt{4x + 5} = 5$.

12. $\sqrt{2x + \sqrt{2x - 4}} = 2$.

12–6. EQUATIONS IN QUADRATIC FORM

Frequently, an equation which is not quadratic in the given unknown may be considered as a quadratic in some expression involving the unknown. Thus, $x^{-4} - 3x^{-2} + 2 = 0$ and $2(x^2 - 2x)^2 - (x^2 - 2x) - 6 = 0$ may be treated as quadratic equations in the expressions x^{-2} and $(x^2 - 2x)$, respectively. This type of situation was met earlier in Examples 12–11 and 12–12. The following examples will further illustrate methods used in solving equations in quadratic form.

Example 12–15. Solve the equation $x^{-4} - 3x^{-2} + 2 = 0$.

Solution: Let $x^{-2} = y$, so that the given equation becomes

$$y^2 - 3y + 2 = 0.$$

Factor, to obtain

$$(y - 1)(y - 2) = 0.$$

Therefore,

$$y = 1 \quad \text{or} \quad y = 2,$$

or

$$x^{-2} = 1 \quad \text{or} \quad x^{-2} = 2.$$

Hence, the solutions are

$$x = \pm 1 \quad \text{and} \quad x = \pm \frac{1}{\sqrt{2}}.$$

Example 12–16. Solve $(x^2 - 2)^2 - 7(x^2 - 2) + 10 = 0$.

Solution: Let $x^2 - 2 = y$, so that we get

$$y^2 - 7y + 10 = 0.$$

Factor, to obtain

$$(y - 2)(y - 5) = 0.$$

Hence,

$$y = 2 \quad \text{or} \quad y = 5,$$

or

$$x^2 - 2 = 2 \quad \text{or} \quad x^2 - 2 = 5.$$

Then,

$$x^2 = 4 \quad \text{or} \quad x^2 = 7,$$

and the roots are

$$x = \pm 2 \quad \text{and} \quad x = \pm \sqrt{7}.$$

Example 12–17. Solve $x^2 + x - 2\sqrt{x^2 + x + 3} = 0$.

Solution: Let $\sqrt{x^2 + x + 3} = y$. Then we can write

$$(x^2 + x + 3) - 2\sqrt{x^2 + x + 3} - 3 = 0, \quad \text{or} \quad y^2 - 2y - 3 = 0.$$

Therefore, $y = -1$ or 3, and $\sqrt{x^2 + x + 3} = -1$ or $\sqrt{x^2 + x + 3} = 3$.

By definition of the radical, $\sqrt{a}$ is a non-negative number. Hence, although $\sqrt{x^2 + x + 3} = -1$ is consistent with the original equation, there are no values of x which satisfy this equation.

Consider, then, $\sqrt{x^2 + x + 3} = 3$. This leads to

$$x^2 + x + 3 = 9, \quad \text{or} \quad x^2 + x - 6 = 0.$$

Hence,

$$x = 2 \quad \text{or} \quad x = -3.$$

Substitution shows that each of these values of x satisfies the original equation.

EXERCISE 12–5

Solve each of the following equations. Check all solutions.

1. $x^4 + x^2 - 12 = 0$.
2. $4x^{-4} - 11x^{-2} - 3 = 0$.
3. $(x^2 + 2)^2 + 3(x^2 + 2) - 4 = 0$.
4. $x^4 - 6x^2 + 8 = 0$.
5. $x^4 - 13x^2 + 36 = 0$.
6. $x^4 - 1 = 0$.
7. $(3x - 4)^2 + 6(3x - 4) + 13 = 0$.
8. $(x^2 + 3x)^2 - 14(x^2 + 3x) + 45 = 0$.
9. $\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) - 48 = 0$.
10. $(x^2 - x)^2 - 20(x^2 - x) + 36 = 0$.

12–7. THE DISCRIMINANT

It will be recalled that the two roots of the general quadratic equation $ax^2 + bx + c = 0$ are

$$(12\text{–}2) \qquad x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$, which appears under the radical sign, is called the *discriminant* of the quadratic polynomial $ax^2 + bx + c$, or the discriminant of equation (12–1).

In what follows we shall assume that a, b, and c are *real*, and we shall make use of the discriminant to determine the character of the roots without actually solving the equation. By inspection of the solutions x_1 and x_2, we reach the following conclusions:

1. If $b^2 - 4ac = 0$, each of the two roots x_1 and x_2 is equal to $-\frac{b}{2a}$, and both roots are thus real.

2. If $b^2 - 4ac$ is positive, then $\sqrt{b^2 - 4ac}$ is real, both roots are real, and they are distinct.

3. If $b^2 - 4ac$ is negative, then $\sqrt{b^2 - 4ac}$ is imaginary, and the roots are distinct complex numbers of the form $\alpha + \beta i$ and $\alpha - \beta i$.

These results may be summarized as follows:

Value of Discriminant	*Character of Roots*
> 0	Real and unequal
$= 0$	Real and equal
< 0	Unequal conjugate complex numbers

Furthermore, if a, b, and c are *rational*, and $b^2 - 4ac$ is a perfect rational square, then the roots are rational; otherwise, they are irrational.

The following examples will illustrate how to determine the character of the roots.

Example 12–18. Determine the character of the roots of

$$2x^2 + 7x - 15 = 0.$$

Solution: Here $a = 2$, $b = 7$, $c = -15$. Hence,

$$b^2 - 4ac = 49 + 120 = 169 = (13)^2.$$

The discriminant is positive, and so the roots are real and unequal. Since the discriminant is a perfect square, the roots are also rational.

Example 12–19. Determine all values of k for which the roots of the equation $kx^2 - 2kx + 4 = 0$ are equal.

Solution: The discriminant must equal zero for the equation to have equal roots. Hence, $b^2 - 4ac = 4k^2 - 16k = 4k(k - 4) = 0$, and so $k = 0$ or $k = 4$. When $k = 0$, the equation is not quadratic. Therefore, the roots are equal only when $k = 4$.

EXERCISE 12–6

Determine the character of the roots of each of the following equations by means of the discriminant.

1. $x^2 + 3x + 4 = 0$. **2.** $x^2 + 8x - 9 = 0$. **3.** $x^2 + 16x - 6 = 0$.

4. $6x^2 - 7x + 3 = 0$. **5.** $x^2 + 10x + 2 = 0$. **6.** $x^2 + 3x + 2 = 0$.

7. $5x^2 + 7x + 2 = 0$. **8.** $4x^2 - 12x + 9 = 0$. **9.** $x^2 - 2x + 1 = 0$.

10. $4x^2 - 8x + 2 = 0$. **11.** $2x^2 - x + 3 = 0$. **12.** $5x^2 + x + 5 = 0$.

13. $4x^2 - 12x - 9 = 0$. **14.** $x^2 + 4x - 18 = 0$. **15.** $x^2 - 4x + 7 = 0$.

12–8. SUM AND PRODUCT OF THE ROOTS

Adding the two roots of the general quadratic equation

$$ax^2 + bx + c = 0,$$

we obtain, by using (12–2),

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}.$$

Also, multiplying these roots, we have

$$x_1x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence, we have, for the sum and product of the roots,

(12–3) $$x_1 + x_2 = -\frac{b}{a},$$

and

(12–4) $$x_1x_2 = \frac{c}{a}.$$

These formulas are used in various ways, for example, in checking roots of a quadratic equation, and in forming an equation if its roots are known.

To find the factored form of the quadratic polynomial $ax^2 + bx + c$, let us make use of the sum and product formulas just found. Solving $x_1 + x_2 = -\frac{b}{a}$ for b, and solving $x_1x_2 = \frac{c}{a}$ for c, we have $b = -a(x_1 + x_2)$ and $c = a(x_1x_2)$. Hence, by substitution, we have

$$\begin{aligned} ax^2 + bx + c &= ax^2 - a(x_1 + x_2)x + a(x_1x_2) \\ &= a(x^2 - (x_1 + x_2)x + (x_1x_2)) \\ &= a(x - x_1)(x - x_2). \end{aligned}$$

Therefore, if x_1 and x_2 are the roots of the quadratic equation $ax^2 + bx + c = 0$, its factored form can be written

$$a(x - x_1)(x - x_2) = 0.$$

Since $a \neq 0$, we may write the equation equivalently as

$$(x - x_1)(x - x_2) = 0.$$

This form or its expansion,

$$x^2 - (x_1 + x_2)x + x_1x_2 = 0,$$

may be used in writing an equation whose roots are known. Example 12–21 indicates the procedure.

Example 12–20. Without solving the equation, find the sum and the product of the roots of the equation $3x^2 - 5x + 2 = 0$.

Solution: In this case, $a = 3$, $b = -5$, and $c = 2$. Then the sum of the roots is $-\frac{b}{a} = \frac{5}{3}$, and the product of the roots is $\frac{c}{a} = \frac{2}{3}$.

Example 12–21. Write a quadratic equation in the form (12–1), given that the roots are $(1 \pm \sqrt{3}\, i)$.

Solution: Since $x_1 = 1 + \sqrt{3}\, i$ and $x_2 = 1 - \sqrt{3}\, i$, we have

$$x_1 + x_2 = (1 + \sqrt{3}\, i) + (1 - \sqrt{3}\, i) = 2,$$

and

$$x_1 x_2 = (1 + \sqrt{3}\, i)(1 - \sqrt{3}\, i) = 1 - 3\, i^2 = 4.$$

Therefore a suitable equation is $x^2 - 2x + 4 = 0$.

Alternate Solution: Writing the desired equation in factored form, we have

$$(x - (1 + \sqrt{3}\, i))(x - (1 - \sqrt{3}\, i)) = 0.$$

Simplifying, we obtain

$$(x - 1 - \sqrt{3}\, i)(x - 1 + \sqrt{3}\, i) = 0,$$

or

$$((x - 1) - \sqrt{3}\, i)((x - 1) + \sqrt{3}\, i) = 0.$$

Hence,

$$(x - 1)^2 + 3 = 0,$$

and, finally,

$$x^2 - 2x + 4 = 0.$$

EXERCISE 12–7

In each of the problems from 1 to 9, find the sum and the product of the roots of the given equation.

1. $x^2 + 2x - 1 = 0$. **2.** $3x^2 - x + 2 = 0$. **3.** $x^2 + 2 = 0$.

4. $6x^2 - 2x + 3 = 0$. **5.** $\frac{x^2}{2} - \frac{3}{4}x + \frac{3}{5} = 0$. **6.** $5x^2 - 6x + 1 = 0$.

7. $5x^2 - 6x - 1 = 0$. **8.** $5x^2 + 6x + 1 = 0$. **9.** $100x^2 - 40x + 17 = 0$.

Form an equation with each of the following pairs of roots.

10. $1, -3$. **11.** $0, +2$. **12.** $-1, 1$. **13.** $3, 6$.

14. $\frac{3}{2}, \frac{1}{3}$. **15.** $2 \pm i\sqrt{5}$. **16.** $\frac{1}{2}(1 \pm i\sqrt{3})$. **17.** $\pm i$.

18. $\sqrt{3}, \sqrt{5}$. **19.** $0, \sqrt{3} - \sqrt{5}$. **20.** $a \pm bi$. **21.** $\sqrt{3} \pm \sqrt{5}\, i$.

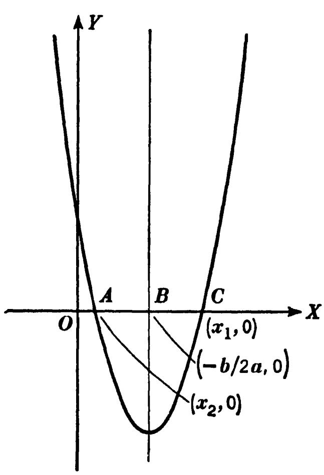

FIG. 12–1.

12–9. GRAPHS OF QUADRATIC FUNCTIONS

To graph any quadratic function of the form $ax^2 + bx + c$, we set $y = ax^2 + bx + c$ and construct a table of values of y corresponding to assigned values of x. The graph is of the type shown in Fig. 12–1 and is called a *parabola*.

As found by the quadratic formula, the two solutions of the general quadratic equation (12–1) are given by

(12–2) $$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Since (12–1) states that $y = 0$ in the equation $y = ax^2 + bx + c$, the solutions x_1 and x_2 are x-intercepts of the curve.

In Fig. 12–1, let A and C be the x-intercept points, and let B be the mid-point of AC. We note that

$$OB = OA + AB,$$

or

$$OB = x_2 + \frac{x_1 - x_2}{2} = \frac{x_1 + x_2}{2} = -\frac{b}{2a}.$$

Therefore, B is the point $\left(-\frac{b}{2a}, 0\right)$.

Now consider the equation $y = k$, which represents a straight line parallel to the x-axis. This line may or may not intersect the parabola, depending on the value of k. Solving the equations $y = k$ and $y = ax^2 + bx + c$ simultaneously, by elimination of y we obtain $ax^2 + bx + c - k = 0$. The roots of this resulting equation are

(12–5) $$x_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4a(c - k)}}{2a}$$

and

$$x_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4a(c - k)}}{2a}.$$

If the value of k is such that $y = k$ intersects the curve in two distinct points, the discriminant in (12–5) is greater than zero, and the roots will be real and distinct. The point on the line $y = k$ with abscissa $x = -\frac{b}{2a}$ is then equidistant from the points of intersection, whose abscissas are x_1 and x_2.

If, on the other hand, the value of k is such that $y = k$ does not intersect the curve, the discriminant in (12–5) is less than zero, and we have a pair of conjugate complex roots, In this case, $-\frac{b}{2a}$ is the real part of these roots.

The points with abscissa $x = -\frac{b}{2a}$ and arbitrary ordinates lie on a vertical line, called the *axis of symmetry*, or simply the *axis* of the curve; the curve is said to be symmetric with respect to the axis.

The point of intersection of the axis and the parabola is called the *vertex* of the parabola. If the coefficient a of the second-degree term of $y = ax^2 + bx + c$ is positive, the vertex is the *lowest point*, and the curve is said to be *concave upward.* If a is negative, the

vertex is the *highest point,* and the curve is said to be *concave downward.*

To find the coordinates of the vertex, we solve simultaneously the equation $x = -\dfrac{b}{2a}$ of the axis and the equation $y = ax^2 + bx + c$ of the parabola. The coordinates of the vertex are thus found to be

$$(12\text{–}6) \qquad x = -\frac{b}{2a} \quad \text{and} \quad y = \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c = -\frac{b^2 - 4ac}{4a}.$$

The vertex may be characterized in another way. Let us demand that a horizontal line $y = k$ intersect the parabola in two coincident points. That is, let us insist that the two roots x_1 and x_2 in (12–5) coincide. Then the discriminant in (12–5) is equal to 0, and $x_1 = x_2 = -\dfrac{b}{2a}$. The value of y corresponding to this value of x is the ordinate of the vertex, as found in (12–6). We say that the line

$$y = -\frac{b^2 - 4ac}{4a}$$

is *tangent* to the parabola at the vertex.

Example 12–22. Graph $x^2 - 6x + 4$.

Solution: Let $y = x^2 - 6x + 4$, assign values to x, and compute the corresponding values of y, as in the accompanying table. The graph is shown in Fig. 12–2. The coordinates of the vertex are

$$x = -\frac{b}{2a} = \frac{6}{2} = 3 \quad \text{and} \quad y = c - \frac{b^2}{4a} = 4 - 9 = -5.$$

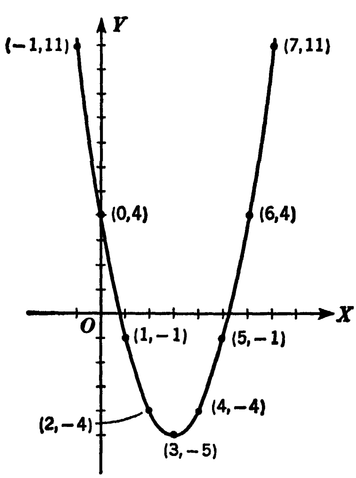

FIG. 12–2.

x	y
-1	11
0	4
1	-1
2	-4
3	-5
4	-4
5	-1
6	4
7	11

Hence, the axis is the line $x = 3$. Since a is positive, the vertex $(3, -5)$ is the lowest point on the curve, and the curve is concave upward.

Example 12–23. Graph $y = x^2 - 6x + 9$ and $y = x^2 - 6x + 14$ relative to the same coordinate system as was used for the graph of $y = x^2 - 6x + 4$.

Solution: Tables similar to that in Example 12–22 but applying to the first two curves are constructed. The three curves are shown in Fig. 12-3.

x	$y = x^2 - 6x + 9$
-1	16
0	9
1	4
2	1
3	0
4	1
5	4
6	9
7	16

x	$y = x^2 - 6x + 14$
-1	21
0	14
1	9
2	6
3	5
4	6
5	9
6	14
7	21

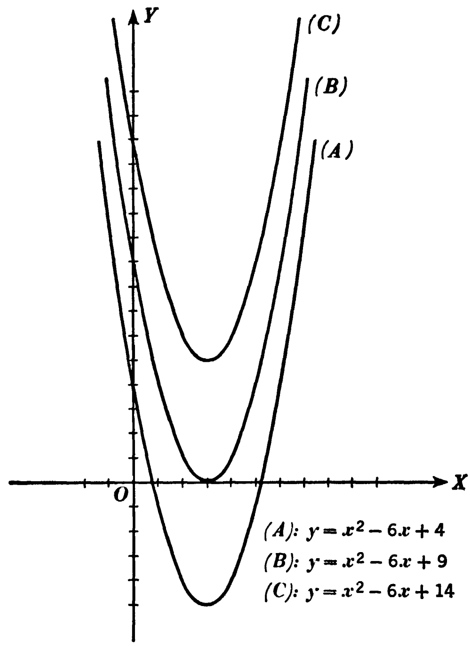

FIG. 12–3.

Curve (A) crosses the x-axis at two points, corresponding to the roots $3 \pm \sqrt{5}$ of $x^2 - 6x + 4 = 0$. Curve (B) is tangent to the x-axis at $(3, 0)$, because both roots of $x^2 - 6x + 9 = 0$ are equal to 3. Curve (C) does not intersect the x-axis, because $x^2 - 6x + 14 = 0$ has imaginary roots.

The reader should relate the discriminants of the quadratics to a study of these graphs.

12–10. QUADRATIC EQUATIONS IN TWO UNKNOWNS

The general equation of the second degree in x and y is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{12–7}$$

where a, b, c, d, e, and f are given real numbers. An equation of this form, in which at least one of the coefficients a, b, and c is different from zero, is called a *quadratic equation in x and y.*

By a solution of such an equation, we mean a pair of real or complex numbers which, when substituted for x and y in (12–7), will reduce the left side of the equation to zero. Usually there are infinitely many pairs of numbers which satisfy the equation.

If $c \neq 0$, (12–7) may be solved for y in terms of x by means of the quadratic formula. Corresponding to each real value assigned to x, we then obtain, in general, two values of y. We then have pairs of numbers (x, y) which, if real, may be plotted in a rectangular-coordinate system. It is shown in analytic geometry that the graph so obtained will be one of a class of curves called *conic sections,* inasmuch as they may be obtained as curves of intersection of a plane and a right circular cone. At this time we shall confine ourselves to merely listing the curves which comprise this class and indicating briefly the form of the quadratic that corresponds to each of the graphs. A more adequate discussion of this subject is given in analytic geometry. However, typical examples of these curves are given here.

1. Parabola. When $A \neq 0$, the equations $y = Ax^2 + Bx + C$ and $x = Ay^2 + By + C$ represent parabolas with vertical and horizontal axes of symmetry, respectively.

2. Circle. When C is positive, the equation $x^2 + y^2 = C$ represents a circle whose center is at the origin and whose radius is $\sqrt{C}$.

3a. Ellipse. When the constants are positive, the equation $Ax^2 + By^2 = C$ represents a curve called an *ellipse.* If $A = B$, the ellipse is a circle.

3b. Point Ellipse. If A and B are positive and $C = 0$, the equation $Ax^2 + By^2 = C$ is satisfied by only one point, namely, the origin. The graph is then said to be a *point ellipse.*

3c. Imaginary Ellipse. If A and B are positive and $C < 0$, there are no (real) points on the graph, and we say that the equation $Ax^2 + By^2 = C$ represents an *imaginary ellipse.*

4a. Hyperbola. When A, B, and C are positive, the equations $Ax^2 - By^2 = C$ and $Ay^2 - Bx^2 = C$ represent *hyperbolas.*

4b. Hyperbola. When $C \neq 0$, the equation $xy = C$ represents a curve called an *equilateral hyperbola.*

5. Pair of Straight Lines. The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents two straight lines, which may be either distinct or coincident, if the left side can be expressed as the product of two real linear factors.

We use the quantity $b^2 - 4ac$, which is usually called the *characteristic* of the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, to determine the nature of the conic corresponding to a particular form of the general quadratic equation. In analytic geometry the following statements are shown to be true:

1. If $b^2 - 4ac = 0$, the conic is a parabola or two real or imaginary parallel lines.
2. If $b^2 - 4ac < 0$, the conic is an ellipse or a point ellipse or an imaginary ellipse.

3. If $b^2 - 4ac > 0$, the conic is a hyperbola or two intersecting lines.

In Section 12–9 the graph of the parabola $y = ax^2 + bx + c$ was discussed. In the following illustrative examples, the procedures for graphs of other quadratic equations are considered.

Example 12–24. Graph $x^2 + y^2 = 9$.

Solution: Set $y = 0$ to obtain the x-intercepts, which are ± 3; and set $x = 0$ to obtain the y-intercepts, which are ± 3. Solve the equation for y, obtaining

$$y = \pm \sqrt{9 - x^2}.$$

Then construct a table of other corresponding values of x and y. To yield real values of y, the numerical value of x cannot exceed 3. As shown in Fig. 12-4, the resulting graph is a circle with center at the origin and radius 3.

x	y
-4	imaginary
-3	0
-2	± 2.24
-1	± 2.83
0	± 3
1	± 2.83
2	± 2.24
3	0
4	imaginary

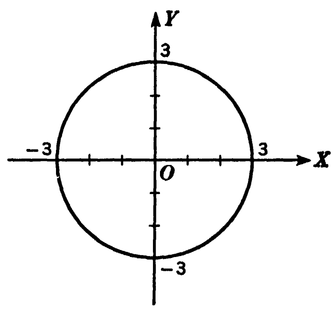

FIG. 12–4.

Example 12–25. Graph $4x^2 + 9y^2 = 36$.

Solution: Set $y = 0$ to obtain the x-intercepts, which are ± 3; and set $x = 0$ to obtain the y-intercepts, which are ± 2. Solve for y and obtain

$$y = \pm \frac{2}{3} \sqrt{9 - x^2}.$$

Construct a table and draw the curve, as shown in Fig. 12-5. This illustrates an ellipse.

x	y
-3	0
-2	± 1.49
-1	± 1.89
0	± 2
1	± 1.89
2	± 1.49
3	0

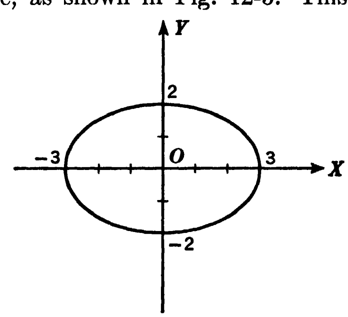

FIG. 12–5.

Note that the numerical value of x must be equal to or less than 3 in order to yield real values of y.

Example 12–26. Graph $4x^2 - 9y^2 = 36$.

Solution: Setting $y = 0$, we find that the x-intercepts are ± 3. Setting $x = 0$, however, results in the equation $y^2 = -4$. Hence the curve has no y-intercepts. Solving the given equation for y, we have

$$y = \pm \frac{2}{3}\sqrt{x^2 - 9}.$$

The accompanying table is constructed.

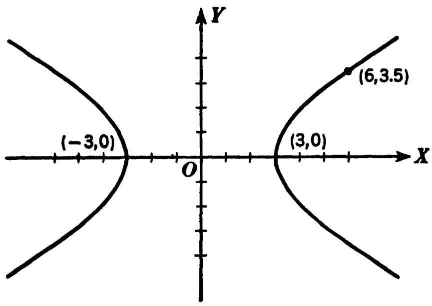

FIG. 12–6.

x	y
-6	± 3.5
-5	± 2.7
-4	± 1.8
-3	0
-2	imaginary
2	imaginary
3	0
4	± 1.8
5	± 2.7
6	± 3.5

Note that in this case the numerical value of x must be equal to or greater than 3 in order to give real values of y. The graph of the given equation is shown in Fig. 12–6. This illustrates a hyperbola.

EXERCISE 12–8

Identify and graph each of the following.

1. $x^2 + y^2 = 25$.
2. $4x^2 + 9x^2 = 36$.
3. $4x^2 - 9y^2 = 0$.
4. $4x^2 - 9y^2 = 36$.
5. $x^2 - 4y^2 = 16$.
6. $x^2 + 9y^2 = 0$.
7. $y = x^2 - 3x + 2$.
8. $xy = -4$.
9. $5x^2 + 9xy = 28y^2$.
10. $2x^2 + y^2 - 4y = 4$.
11. $5xy = 2x + y$.
12. $3x^2 - 4xy + 2y^2 - 6x + 3y = 7$.
13. $x^2 + xy - 2y^2 + 3y - 1 = 0$.
14. $4x^2 - 4xy + y^2 - 2x + 4y - 12 = 0$.
15. $xy + y^2 - y - 2x - 2 = 0$.
16. $x^2 - 3x - 3y^2 + 18y = 27$.
17. $2x^2 - xy - 28y^2 = 0$.
18. $4x^2 + 4xy - 3y^2 + 4x + 10y = 3$.
19. $9x^2 - 24xy + 16y^2 + 3x - 4y = 6$.
20. $4x^2 + 3xy + 4y^2 - 8x - 8y = 24$.

12–11. GRAPHICAL SOLUTIONS OF SYSTEMS OF EQUATIONS INVOLVING QUADRATICS

In Chapter 9 we solved systems of two or more linear equations both algebraically and graphically. Frequently, however, simultaneous systems include one or more equations of the second or higher degree. We have, therefore, to consider the problem of finding systems of values of the unknowns x and y that satisfy two equa-

tions, one of which is quadratic and the other of which is linear or quadratic.

We shall begin by illustrating some graphical solutions of several types of systems. The graphical method yields only the real solutions of a system, but it may prove advantageous in suggesting solutions and interpreting results. In general, this method yields at best only approximate solutions. The graphs should be drawn as accurately as possible.

Example 12–27. Solve graphically the system

$$\begin{cases} x^2 - 8x + 3y = 0, \\ x - 3y + 6 = 0. \end{cases}$$

Solution: Solving each equation for y in terms of x, we have

$$y = \frac{8x - x^2}{3} \quad \text{and} \quad y = \frac{x + 6}{3}.$$

Construct tables of values, and draw both graphs, using the same coordinate system, as shown in Fig. 12-7.

x	$y = \frac{8x - x^2}{3}$
-1	-3
0	0
1	7/3
2	4
3	5
4	16/3
5	5
6	4
7	7/3
8	0
9	-3

x	$y = \frac{x + 6}{3}$
-6	0
0	2

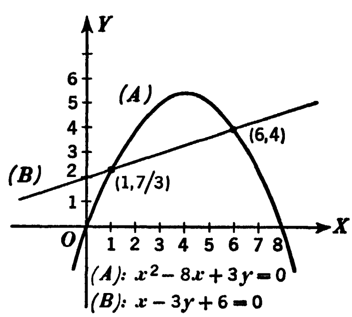

FIG. 12–7.

The line and the parabola are seen to intersect at the points (1, 7/3) and (6, 4). It follows that these points represent common real solutions, possibly only approximate, of the system. A check by substitution shows that the real solutions are, in fact, $x = 1$, $y = 7/3$; and $x = 6$, $y = 4$.

The solution of Example 12–27 suggests a procedure for finding graphical solutions of quadratic equations in one unknown.

Example 12–28. Solve the equation $x^2 - x - 2 = 0$ graphically.

Solution: Since $x^2 = x + 2$, both sides of this equation may be set equal to y. Thus, the original equation is replaced by the system

$$\begin{cases} y = x^2, \\ y = x + 2. \end{cases}$$

x	$y = x^2$
0	0
± 1	1
± 2	4
± 3	9
± 4	16

x	$y = x + 2$
0	2
-2	0

From the accompanying tables of values of x and y, the graphs shown in Fig. 12-8 are constructed. The graphs show that the line and the parabola intersect at the points $(-1, 1)$ and $(2, 4)$. Since these points are common to both graphs, their abscissas must satisfy the equation

$$x^2 = x + 2.$$

Hence, the required roots of the original equation $x^2 - x - 2 = 0$ are $x = -1$ and $x = 2$.

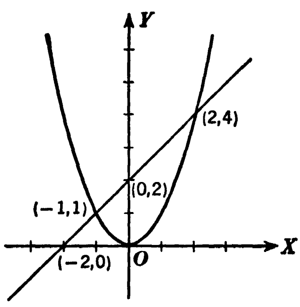

FIG. 12–8.

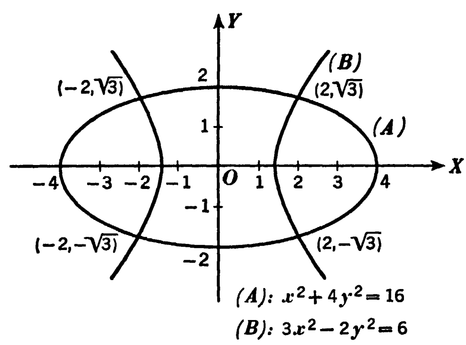

FIG. 12–9.

Example 12–29. Solve graphically the system

$$\begin{cases} x^2 + 4y^2 = 16, \\ 3x^2 - 2y^2 = 6. \end{cases}$$

Solution: From the first given equation,

$$4y^2 = 16 - x^2,$$

and

$$y = \pm \frac{1}{2} \sqrt{16 - x^2}.$$

From the second equation,

$$2y^2 = 3x^2 - 6,$$

and

$$y = \pm \sqrt{\frac{3x^2 - 6}{2}}.$$

The necessary tables are given here, and the graphs are shown in Fig. 12-9.

x	$y = \pm\frac{1}{2}\sqrt{16 - x^2}$	x	$y = \pm\sqrt{\frac{3x^2 - 6}{2}}$
0	± 2	0	imaginary
1	$\pm\frac{1}{2}\sqrt{15}$	± 1	imaginary
2	$\pm\sqrt{3}$	$\pm\sqrt{2}$	0
3	$\pm\frac{1}{2}\sqrt{7}$	2	$\pm\sqrt{3}$
4	0	3	$\pm\sqrt{10.5}$
-1	$\pm\frac{1}{2}\sqrt{15}$	4	$\pm\sqrt{21}$
-2	$\pm\sqrt{3}$	-2	$\pm\sqrt{3}$
-3	$\pm\frac{1}{2}\sqrt{7}$	-3	$\pm\sqrt{10.5}$
-4	0	-4	$\pm\sqrt{21}$

The ellipse (A) and the hyperbola (B) are seen to intersect in the following four distinct points:

$$(2, \sqrt{3}), \quad (2, -\sqrt{3}), \quad (-2, \sqrt{3}), \quad (-2, -\sqrt{3}).$$

These values of x and y already appear in the tables used for constructing the graphs, and need not be checked by substitution in the original equations. However, such checking is usually desirable.

EXERCISE 12–9

Solve each of the following systems of equations graphically.

1. $\begin{cases} x - 2y + 3 = 0, \\ x^2 = 3y. \end{cases}$ **2.** $\begin{cases} x + y = 4, \\ y^2 = 2x. \end{cases}$ **3.** $\begin{cases} x - y + 1 = 0, \\ x^2 + y^2 = 25. \end{cases}$

4. $\begin{cases} y^2 = 3x, \\ 3x + y = 6. \end{cases}$ **5.** $\begin{cases} x + y = 6, \\ x^2 = y. \end{cases}$ **6.** $\begin{cases} 3x + 2y = 6, \\ xy = -12. \end{cases}$

7. $\begin{cases} 2x - y = 4, \\ xy = 6. \end{cases}$ **8.** $\begin{cases} x^2 - y^2 = 16, \\ x + 3y = 4. \end{cases}$ **9.** $\begin{cases} 8x + 3y = 25, \\ 4x^2 + y^2 = 25. \end{cases}$

10. $\begin{cases} x^2 - 2y^2 - 4 = 0, \\ x^2 - 9y = 0. \end{cases}$ **11.** $\begin{cases} x^2 + y^2 = 13, \\ x^2 = 12y. \end{cases}$ **12.** $\begin{cases} 9x^2 + 4y^2 = 36, \\ x^2 + y^2 = 81. \end{cases}$

13. $\begin{cases} 9x^2 + 25y^2 = 225, \\ x^2 + y^2 = 4. \end{cases}$ **14.** $\begin{cases} x^2 + y^2 = 20, \\ y^2 - x^2 = 12. \end{cases}$ **15.** $\begin{cases} x^2 + y^2 = 20, \\ 4x^2 + 9y^2 = 100. \end{cases}$

16. $\begin{cases} x + y = 0, \\ x^2 + y^2 = 8. \end{cases}$

12–12. ALGEBRAIC SOLUTIONS OF SYSTEMS INVOLVING QUADRATICS

As in the case with linear equations discussed in Section 9-1, it may happen that in a system of equations involving quadratics

part of the graph of one equation coincides with part of the graph of the other. Such a condition gives rise to infinitely many solutions. Usually, however, there are only a finite number of points of intersection of the graphs corresponding to the given equations, and the algebraic problem consists of finding the pairs of numbers (x, y) which satisfy both equations. We can say in this case that two simultaneous equations in x and y, of degrees m and n, respectively, can have at most mn solutions. Thus, a system of one linear and one quadratic equation can have at most two solutions, and a system of two quadratics can have at most four solutions.

When a system consists of two quadratic equations, the algebraic solution usually leads to a fourth-degree equation in one of the unknowns. Since we have not presented a general method of solving a fourth-degree equation, we shall consider here only systems whose solutions can be effected by the theory of quadratic equations. The methods of procedure in some of the more important types are shown in the following three cases:

Case 1. *One Linear and One Quadratic Equation.* A system of this type can always be solved by the method of elimination by substitution.

Example 12–30. Solve the system

$$\begin{cases} x - 3y + 6 = 0, \\ x^2 - 8x + 3y = 0. \end{cases}$$

Solution: Solve the linear equation for y in terms of x, obtaining

$$y = \frac{x + 6}{3}.$$

Substitution for y in the quadratic equation yields

$$x^2 - 8x + 3\left(\frac{x + 6}{3}\right) = 0.$$

Collecting terms gives

$$x^2 - 7x + 6 = 0.$$

The roots of this equation are $x = 1$ and $x = 6$. Substituting these values in the linear equation, we obtain $y = 7/3$ and $y = 4$. Hence, the solutions are

$$x = 1,\ y = 7/3; \quad \text{and} \quad x = 6,\ y = 4.$$

These values can readily be verified as solutions of the given system. Note that they correspond to the coordinates of the points of intersection in Fig. 12-7.

Example 12–31. Solve the system

$$\begin{cases} x + 2y + 4 = 0, \\ x^2 + 4y^2 - 2x - 3 = 0. \end{cases}$$

Solution: Solve the linear equation for $2y$, to obtain

$$2y = -(x+4).$$

Substitute in the quadratic and collect terms. We then have

$$x^2 + (x+4)^2 - 2x - 3 = 0,$$

or

$$2x^2 + 6x + 13 = 0.$$

One solution is

$$x = \frac{-3 + i\sqrt{17}}{2}, \quad y = -\frac{5 + i\sqrt{17}}{4}.$$

The other solution is

$$x = \frac{-3 - i\sqrt{17}}{2}, \quad y = -\frac{5 - i\sqrt{17}}{4}.$$

Since these values are imaginary, the graphs of the two given equations do not intersect.

Case 2. *Two Equations of the Form* $ax^2 + by^2 = c$. When the system consists of two equations containing only squared terms in each unknown, it can be solved for x^2 and y^2 by the methods used for linear systems in Section 9–2.

Example 12–32. Solve the system

$$\begin{cases} x^2 + 2y^2 = 17, \\ 2x^2 - y^2 = 14. \end{cases}$$

Solution: To eliminate y^2, multiply the second equation by 2 and add the two equations, as follows:

$$\begin{array}{r} x^2 + 2y^2 = 17 \\ 4x^2 - 2y^2 = 28 \\ \hline 5x^2 = 45. \end{array}$$

Solving for x, we have

$$x = \pm 3.$$

Now substitute 9 for x^2 in the first of the original equations. Then

$$2y^2 = 17 - 9 = 8,$$

or

$$y = \pm 2.$$

Hence, we have the following four solutions:

$$(3, 2), \quad (3, -2), \quad (-3, 2), \quad (-3, -2).$$

These may be written $(3, \pm 2)$, $(-3, \pm 2)$.

Case 3. *Two Equations of the Form* $ax^2 + bxy + cy^2 = d$. If the system is of this type, the solution is effected by elimination of the constant term. The resulting equation is then solved for one unknown in terms of the other. This procedure gives us two linear equations in x and y which may be combined with either of the given quadratic equations to form two systems of the type considered in case 1.

Example 12–33. Solve the system

$$\begin{cases} x^2 - xy + 2y^2 = 1, \\ 2x^2 - 2xy + 8y^2 = 3. \end{cases}$$

Solution: Multiply the first equation by 3 and subtract the second given equation from the new equation, as follows:

$$\begin{array}{r} 3x^2 - 3xy + 6y^2 = 3 \\ \underline{2x^2 - 2xy + 8y^2 = 3} \\ x^2 - xy - 2y^2 = 0. \end{array}$$

Factor, to obtain

$$(x + y)(x - 2y) = 0.$$

Hence,

$$x + y = 0 \quad \text{or} \quad x - 2y = 0.$$

We may combine each of these two equations with the first given equation to form the following two systems:

$$\begin{cases} x^2 - xy + 2y^2 = 1, \\ x + y = 0; \end{cases}$$

and

$$\begin{cases} x^2 - xy + 2y^2 = 1, \\ x - 2y = 0. \end{cases}$$

We then proceed by the method for case 1.

The solutions of the given system consist of the solutions of these two systems. Hence, we have

$$x = 1/2, y = -1/2; \quad x = -1/2, y = 1/2;$$
$$x = 1, y = 1/2; \quad x = -1, y = -1/2.$$

Occasionally, another method is effective in connection with systems described under case 3. The following example illustrates this method.

Example 12–34. Solve the system

$$\begin{cases} x^2 + 9y^2 = 37, \\ xy = 2. \end{cases}$$

Solution: Multiply the second equation by 6, to obtain $6xy = 12$. Add this to the first equation, to obtain

$$x^2 + 6xy + 9y^2 = 49.$$

Also subtract $6xy = 12$ from the first equation to obtain

$$x^2 - 6xy + 9y^2 = 25.$$

The left side of each of these new equations is a perfect square. We chose the multiplier of xy, which is 6 in this case, so as to obtain perfect squares. We now have the system

$$\begin{cases} (x + 3y)^2 = 49, \\ (x - 3y)^2 = 25. \end{cases}$$

Hence,

$$x + 3y = 7 \quad \text{or} \quad x + 3y = -7.$$

Also,

$$x - 3y = 5 \quad \text{or} \quad x - 3y = -5.$$

We now solve the following four systems of linear equations:

$$\begin{cases} x + 3y = 7, \\ x - 3y = 5; \end{cases} \quad \begin{cases} x + 3y = 7, \\ x - 3y = -5; \end{cases} \quad \begin{cases} x + 3y = -7, \\ x - 3y = 5; \end{cases} \quad \begin{cases} x + 3y = -7, \\ x - 3y = -5. \end{cases}$$

An equation in x and y is said to be *symmetric in x and y* if the equation is unchanged when x and y are interchanged. When a system consists of two quadratic equations both of which are symmetric in x and y, the substitutions $x = u + v$, $y = u - v$ will give an equivalent system which may in some cases be solved by previous methods.

EXERCISE 12–10

In each of the problems from 1 to 21, solve the given system of equations algebraically.

1. $\begin{cases} x - 2y + 3 = 0, \\ x^2 = 3y. \end{cases}$ **2.** $\begin{cases} x + y = 4, \\ y^2 = 2x. \end{cases}$ **3.** $\begin{cases} x - y + 1 = 0, \\ x^2 + y^2 = 25. \end{cases}$

4. $\begin{cases} y^2 = 3x, \\ 3x + y = 6. \end{cases}$ **5.** $\begin{cases} 3x + 2y = 6, \\ xy + 12 = 0. \end{cases}$ **6.** $\begin{cases} 2x - y = 4, \\ xy = 6. \end{cases}$

7. $\begin{cases} x^2 - y^2 = 16, \\ x + 3y = 4. \end{cases}$ **8.** $\begin{cases} 8x + 3y = 25, \\ 4x^2 + y^2 = 25. \end{cases}$ **9.** $\begin{cases} y - x = 2, \\ x^2 + y^2 - 2x - 4y = 20. \end{cases}$

10. $\begin{cases} x^2 - 2y^2 = 4, \\ x^2 - 9y = 0. \end{cases}$ **11.** $\begin{cases} x^2 + y^2 = 10, \\ x^2 = 9y. \end{cases}$ **12.** $\begin{cases} (x - 3)^2 + (y - 1)^2 = 16, \\ (x - 1)^2 + (y - 1)^2 = 12. \end{cases}$

13. $\begin{cases} 9x^2 + 4y^2 = 36, \\ x^2 + y^2 = 81. \end{cases}$ **14.** $\begin{cases} 9x^2 + 25y^2 = 225, \\ x^2 + y^2 = 4. \end{cases}$ **15.** $\begin{cases} x^2 + y^2 = 20, \\ y^2 - x^2 = 12. \end{cases}$

16. $\begin{cases} xy = 2, \\ x^2 - y^2 = 3. \end{cases}$ **17.** $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{2}, \\ x + y = 3. \end{cases}$ **18.** $\begin{cases} x^2 + 3xy = 10, \\ xy = 3. \end{cases}$

19. $\begin{cases} x^2 + 3xy = 28, \\ xy + 4y^2 = 8. \end{cases}$ **20.** $\begin{cases} 3x^2 + y^2 = 28, \\ 4x^2 - xy + y^2 = 40. \end{cases}$ **21.** $\begin{cases} xy + y^2 = 12, \\ xy = 2x^2 - 24. \end{cases}$

22. Complete the solution of the system in Example 12–34.

In each of the problems from 23 to 26, solve the given system by the method of Example 12–34.

23. $\begin{cases} x^2 + y^2 = 50, \\ xy = 25. \end{cases}$ **24.** $\begin{cases} x^2 + 4y^2 = 13, \\ xy = 3. \end{cases}$

25. $\begin{cases} x^2 + y^2 = 25, \\ xy = 12. \end{cases}$ **26.** $\begin{cases} x^2 + y^2 = 144, \\ xy = 56. \end{cases}$

Solve each of the following symmetric systems.

27. $\begin{cases} x^2 + y^2 = 4, \\ xy + x + y = 10. \end{cases}$ **28.** $\begin{cases} x^2 + y^2 - x - y = 2, \\ xy + 3x + 3y = 2. \end{cases}$

29. $\begin{cases} x^2/y + y^2/x = 56, \\ x + y = 2. \end{cases}$ **30.** $\begin{cases} x^2 + y^2 = 13, \\ 3x^2 + 2xy + 3y^2 = 42. \end{cases}$

12–13. EXPONENTIAL AND LOGARITHMIC EQUATIONS

An equation in which the unknown occurs in an exponent is called an *exponential equation.* Such an equation is usually solved by taking the logarithm of each side and solving the resulting equation. When this latter equation is in linear or quadratic form, it may be solved by preceding methods.

Example 12–35. Solve for x: $5^{x+3} = 625$.

Solution: Write the equation in the form

$$5^{x+3} = 5^4.$$

This equation is satisfied if and only if $x + 3 = 4$, that is, $x = 1$.

Example 12–36. Solve the equation $2^{3x+1} = 3^{4x}$.

Solution: Taking the logarithm of each side to the base 10, we get

$$\log (2^{3x+1}) = \log (3^{4x}),$$

or

$$(3x + 1) \log 2 = 4x \log 3.$$

Therefore,

$$x = \frac{\log 2}{4 \log 3 - 3 \log 2}.$$

From Table III, $\log 2 = 0.3010$ and $\log 3 = 0.4771$. Substituting these values, we have

$$x = \frac{0.3010}{4 \cdot 0.4771 - 3 \cdot 0.3010},$$

or

$$x = \frac{0.3010}{1.0054} = 0.2994.$$

Example 12–37. Solve for x: $\log (x + 1) = 1 - \log (3x + 2)$.

Solution: Collecting terms containing logarithms on one side and writing that member as a single logarithm, we have

$$\log (x + 1) + \log (3x + 2) = 1,$$

or

$$\log (x + 1) (3x + 2) = 1.$$

Writing the members in exponential form, we have

$$(x + 1) (3x + 2) = 10^1 = 10.$$

This equation reduces to

$$3x^2 + 5x - 8 = 0.$$

Solving for x, we find that $x = 1$ or $x = -\frac{8}{3}$.

Checking, we find that $x = 1$ satisfies the original equation, whereas $x = -\frac{8}{3}$ gives rise to logarithms of negative numbers. Since negative numbers do not have real logarithms, this latter value of x is not to be used.

Example 12–38. Solve $y = \log_e (x + \sqrt{1 + x^2})$ for x in terms of y.

Solution: Write the given equation in exponential form, obtaining

$$e^y = x + \sqrt{1 + x^2}.$$

Transpose and square to remove the radical. The result is

$$e^{2y} - 2xe^y - 1 = 0.$$

Solving this equation for x, we have

$$x = \frac{e^{2y} - 1}{2e^y}.$$

EXERCISE 12-11

Solve each of the following equations for the unknown x.

1. $2^x = 64$.
2. $4^{x+1} = 256$.
3. $3^{3x+1} = 243$.
4. $10^x = 0.0001$.
5. $4^x = 24$.
6. $2^{4x} = 2^{3x-1}$.
7. $5^x = 15$.
8. $3^x = 17$.
9. $3(2^x) = 6^x$.
10. $(3^x)(2^x) = 36$.
11. $5^{1/x} = 2.403$.
12. $\log_x 8 = 0.4136$.
13. $\log_x 2 = 0.6932$.
14. $x^{3.146} = 0.04681$.
15. $(1.5)^x = 32$.
16. $5.03 = (3.17)^{1/(x-1)}$.
17. $3^{x+2} = 2(5^x)$.
18. $\log_2 x = 3$.
19. $\log(3x - 5) = 3 - \log 7$.
20. $\log(4x - 1) = 1 - \log(6x + 2)$.
21. $x^{\log(x^3)} = 1000$.
22. $\log_2(x - 1) + \log_2(x + 3) = 3$.
23. $e^{2x} = 4.83$.
24. $e^{(x^2)} = 9.436$.
25. $4e^{x+1} = 7$.
26. $e^{x^2+2x-2} = 16$.
27. $ac + bc^{x+1} = 1$.
28. $y = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$.
29. $y = \frac{e^x + e^{-x}}{2}$.
30. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$.
31. $e^{4x} - e^{2x} - 10 = 0$.
32. $e^x + 4e^{-x} = 5$.

12-14. GRAPHS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

The graph of $y = \log x$ is shown in Fig. 12-10. By assigning values to x, one finds corresponding values of y from Table III. A few pairs of values are shown in the accompanying tabulation.

x	y
0.1	-1
0.2	-0.70
0.3	-0.52
0.5	-0.30
1	0
2	0.30
3	0.48
4	0.60
10	1.

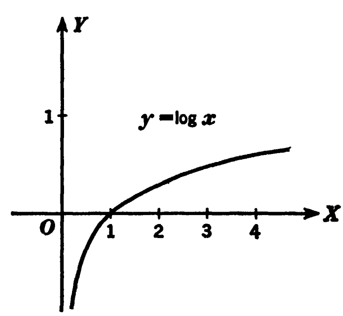

FIG. 12-10.

The graph of $y = e^x$ may be obtained from a table of exponential functions. Here, however, we shall proceed as follows: Take the logarithm of each side to the base 10, to obtain

$$\log y = x \log e.$$

Prepare the accompanying table, and construct the graph in Fig. 12–11.

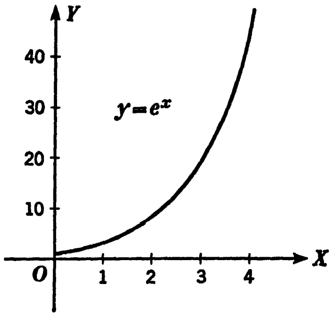

FIG. 12–11.

x	$\log e$	$x \log e$	y
0	0.434	0	1.0
1	0.434	0.434	2.7
2	0.434	0.868	7.4
3	0.434	1.302	20.0
4	0.434	1.736	54.5

EXERCISE 12–12

Graph each of the following.

1. $y = \log_7 x$.
Hint: $\log_7 x = \dfrac{\log_{10} x}{\log_{10} 7}$.

2. $x = \log_7 y$.

3. $y = 3^x$.

4. $y = 10^{-x^2}$.

5. $y = e^{\frac{x^2}{2}}$.

6. $y = 16(5^x)$.

7. $y = 3 + 4(2^x)$.

8. $y = e^{-x}$.

9. $y = \log_e x$.

10. $y = 100e^{0.05x}$.

11. $y = 7^{3x-1}$.

12. $y = 3.9^{2x-3}$.

13 Theory of Equations

13–1. INTRODUCTORY REMARKS

With the ever-increasing importance of mathematics in engineering and the physical sciences, problems are constantly occurring that involve the solution of equations. Often these equations are of the simple algebraic or trigonometric types which we have already learned to solve. There are many other problems, however, which require the solution of equations of higher degree than the second and of some types of somewhat more complicated transcendental equations. Equations of the third and fourth degree can be solved by methods analogous to those which we used for quadratic equations. Because of their complexity, however, these methods are seldom used. It has been proved that no such procedures exist for equations of degree higher than the fourth.

In this chapter we shall consider various properties of polynomial equations in general. Some of these properties will be of considerable use in later studies of mathematics, while others are considered here merely for the aid they give us in determining roots of equations.

13–2. SYNTHETIC DIVISION

A simplification of the ordinary method of long division, called *synthetic division,* will be presented here. This abbreviated method not only enables us to quickly find the quotient and remainder when a polynomial $f(x) = a_0x^n + a_1 x^{n-1} + \cdots + a_n$ is divided by a binomial of the form $x - r$, but also affords a simple process for substituting values of the variable into a polynomial.

We shall divide $2x^3 - 9x^2 + 13x + 5$ by $x - 3$ to illustrate the procedure in synthetic division as compared with that of long division considered in Section 1–19.

By long division we have:

$$
\begin{array}{rrrr|l}
2x^3 & -\ 9x^2 & +\ 13x & +\ \ 5 & x - 3 \\ \cline{5-5}
\underline{2x^3} & \underline{-\ 6x^2} & & & 2x^2 - 3x + 4 \\ \cline{5-5}
 & -\ 3x^2 & +\ 13x & & \\
 & \underline{-\ 3x^2} & \underline{+\ \ 9x} & & \\
 & & 4x & +\ \ 5 & \\
 & & \underline{4x} & \underline{-\ 12} & \\
 & & & +\ 17 &
\end{array}
$$

Thus, the quotient is $2x^2 - 3x + 4$, and the remainder is 17.

Since like powers of x are written in the same vertical column, the work may be shortened by writing only the coefficients, as in the following schematic arrangement:

$$
\begin{array}{rrrr|lll}
2 & -9 & +13 & +5 & 1 & -3 & \\ \cline{5-7}
\underline{2} & \underline{-6} & & & 2 & -3 & +4 \\ \cline{5-7}
 & -3 & +13 & & & & \\
 & \underline{-3} & \underline{+\ 9} & & & & \\
 & & +\ 4 & +\ 5 & & & \\
 & & \underline{+\ 4} & \underline{-12} & & & \\
 & & & +17 & & &
\end{array}
$$

Next, we note that the first term in the divisor $x - r$ need not be written, since the divisor is always linear, and the coefficient of x in it is always unity. Moreover, it is not necessary to write the first term in each row that is to be subtracted, since its coefficient is always the same as that of the term directly above. Also, only the first term of each partial remainder needs to be written down, for the second term is the same as the term directly above it in the first row. Finally, the coefficients in the quotient need not be written, since these are precisely the leading coefficient in the dividend and the remaining partial remainders, excepting the last. Hence, we may indicate the process in the following way:

$$
\begin{array}{rrrr|l}
2 & -9 & +13 & +5 & \underline{-3} \\
 & \underline{-6} & & & \\
 & -3 & & & \\
 & & \underline{+\ 9} & & \\
 & & 4 & & \\
 & & & \underline{-12} & \\
 & & & +17 &
\end{array}
$$

This scheme can be written compactly as follows:

$$\begin{array}{rrrr|l} 2 & -9 & 13 & 5 & \underline{-3} \\ & -6 & 9 & 12 & \\ \hline 2 & -3 & 4 & 17 & \end{array}$$

If we replace -3 by $+3$ in the divisor and *add* the partial products in the second row instead of subtracting them, we obtain the same result. The synthetic division then takes the following form:

$$\begin{array}{rrrr|l} 2 & -9 & 13 & 5 & \underline{3} \\ & 6 & -9 & 12 & \\ \hline 2 & -3 & 4 & 17 & \end{array}$$

Here the numbers 2, -3, and 4 in the third row are the coefficients of the quotient, and the last number 17 is the remainder.

We can now outline the procedure for synthetic division. Note that in every step of the procedure, immediate reference is made to the illustrative example.

To divide $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ by $x - r$, first arrange $f(x)$ in descending powers of x, writing zero for the coefficient of any missing power of x. Then arrange the numbers involved in the process in three rows, as shown in the following steps:

Step 1. In the first row, write the coefficients in $f(x)$ in order, as $a_0, a_1, \cdots a_n$. At the right, put the constant term of the divisor with its sign changed. We have then

Example

$$\begin{array}{lllll|l} a_0 & a_1 & a_2 & \cdots & a_n & \underline{r} \end{array} \qquad \begin{array}{rrrr|l} 2 & -9 & 13 & 5 & \underline{3} \end{array}$$

Step 2. Bring down the first coefficient a_0 of $f(x)$ into the first place of the third row. Thus, we have

Example

$$\begin{array}{lllll|l} a_0 & a_1 & a_2 & \cdots & a_n & \underline{r} \\ \hline a_0 & & & & & \end{array} \qquad \begin{array}{rrrr|l} 2 & -9 & 13 & 5 & \underline{3} \\ \hline 2 & & & & \end{array}$$

Step 3. Multiply a_0 by r, and write the product a_0r in the second row under a_1. Bring down the sum of a_1 and a_0r into the third row. Thus, we now have

Example

$$\begin{array}{lllll|l} a_0 & a_1 & a_2 & \cdots & a_n & \underline{r} \\ & a_0r & & & & \\ \hline a_0 & (a_0r + a_1) & & & & \end{array} \qquad \begin{array}{rrrr|l} 2 & -9 & 13 & 5 & \underline{3} \\ & 6 & & & \\ \hline 2 & -3 & & & \end{array}$$

In the example, multiply 2 by 3 and write the product 6 in the second row below -9. Then add -9 and 6, writing the sum -3 in the third row.

Step 4. Multiply $a_0r + a_1$ by r, place the product in the second row under a_2, and add. Continue this process until finally a product has been added to a_n.

The complete solution of the illustrative example follows:

$$\begin{array}{rrrr|l} 2 & -9 & 13 & 5 & \underline{3} \\ & 6 & -9 & 12 & \\ \hline 2 & -3 & 4 & 17 & \end{array}$$

In the last operations, -3 is multiplied by 3, and the product -9 is written below 13. The sum of 13 and -9 equals 4. Finally the product of 4 and 3, or 12, is written below 5 and added to $a_n = 5$ to give 17.

Step 5. When the process is completed, the last number in the third row directly below a_n is the remainder. The other numbers in this row, read from left to right, are the coefficients of powers of x in the quotient arranged in descending order.

The entire synthetic process of dividing a polynomial $f(x)$ by $x - r$, although it is somewhat complex notationally, can be conveniently exhibited as follows:

$$\begin{array}{cccccc|l} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & \underline{r} \\ & b_0r & b_1r & \cdots & b_{n-2}r & b_{n-1}r & \\ \hline b_0 & b_1 & b_2 & \cdots & b_{n-1} & R & \end{array}$$

Here the expressions for the coefficients $b_0, b_1, b_2, \cdots, b_{n-1}$ of the powers of x in the quotient are $b_0 = a_0$, $b_1 = a_0 r + a_1$, $b_2 = a_0 r^2 + a_1 r + a_2 \cdots$, $b_{n-1} = a_0 r^{n-1} + a_1 r^{n-2} + \cdots + a_{n-1}$. Hence, the quotient may be written

$$q(x) = b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1}. \tag{13–1}$$

Also, the remainder assumes the form

$$R = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n. \tag{13–2}$$

The expression for R is precisely the result of substituting r for x in $f(x)$. In other words

$$R = f(r). \tag{13–3}$$

Finally, if we subtract the product of $x - r$ and $q(x)$ from $f(x)$, we obtain the remainder $R = f(r)$. Or, if we transpose the product, we have the usual statement found in discussions of division. That is,

$$f(x) = (x - r) \cdot q(x) + f(r). \tag{13–4}$$

The following examples will further illustrate the process of synthetic division.

Example 13-1. Divide $3x^4 - 4x^2 + x - 2$ by $x + 2$.

Solution: Since $x - r = x + 2$, we have $r = -2$. Writing zero for the coefficient of the missing power x^3, we have the following result:

$$\begin{array}{rrrrr|l} 3 & 0 & -4 & 1 & -2 & \underline{-2} \\ & -6 & 12 & -16 & 30 & \\ \hline 3 & -6 & 8 & -15 & 28 & \end{array}$$

Note that the first coefficient 3 is brought down into the first place of the third row. Next 3 is multiplied by -2, and the product, which is -6, is written in the second row under 0. The sum of 0 and -6, or -6, is written in the third row directly below 0. Proceeding in this way, we find that the quotient is $3x^3 - 6x^2 + 8x - 15$ and the remainder is 28.

Example 13-2. Given $f(x) = x^3 - 2x^2 + 5x - 4$, find
a) $f(-1)$; *b)* $f(1)$; *c)* $f(3)$.

Solution: By (13-3), $f(r)$ equals the remainder obtained in the division of $f(x)$ by $x - r$. Hence, we have the following results:

a)
$$\begin{array}{rrrr|l} 1 & -2 & 5 & -4 & \underline{-1} \\ & -1 & 3 & -8 & \\ \hline 1 & -3 & 8 & -12 & \end{array}$$

Therefore, $f(-1) = -12$.

b)
$$\begin{array}{rrrr|l} 1 & -2 & 5 & -4 & \underline{1} \\ & 1 & -1 & 4 & \\ \hline 1 & -1 & 4 & 0 & \end{array}$$

Therefore, $f(1) = 0$.

c)
$$\begin{array}{rrrr|l} 1 & -2 & 5 & -4 & \underline{3} \\ & 3 & 3 & 24 & \\ \hline 1 & 1 & 8 & 20 & \end{array}$$

Therefore, $f(3) = 20$.

EXERCISE 13-1

In each of the problems from 1 to 15, divide the first function by the second, to find the quotient and the remainder, by using synthetic division.

1. $x^2 - 8x + 7$, $x - 1$.
2. $x^3 - x^2 - 8x + 6$, $x - 3$.
3. $x^2 - 5x + 6$, $x - 4$.
4. $x^3 - 3x^2 + 6x - 6$, $x - 3$.
5. $x^4 - 3x^2 - 6x + 6$, $x + 3$.
6. $x^3 - 3x^2 + 6x - 24$, $x - 4$.
7. $2x^4 - x^3 - 6x^2 + 4x - 8$, $x - 2$.
8. $2x^3 + 3x^2 + 4$, $x + 2$.
9. $x^3 + 3x^2 - 2x - 5$, $x - 2$.
10. $2x^5 - 3x^3 + 2x + 1$, $x + 2$.
11. $x^4 + x^3 - 59x^2 - 69x + 630$, $x^2 - x - 42$. (Hint: Factor the divisor and divide successively by each factor.)

12. $x^4 - 3x^3 + 3x^2 - 3x + 2, x^2 - 3x + 2$.
13. $x^5 + 2x^4 - 21x + 18, x^2 + 2x - 3$.
14. $a^n - 1, a - 1$.
15. $x^n - y^n, x - y$.

For each of the following polynomial functions, find the indicated values by the method of synthetic division:

16. $f(x) = 3x^3 - 7x^2 - 5x + 6$. Find $f(1)$ and $f(-4)$.
17. $f(x) = x^4 - 2x^3 + 2x^2 - 5x + 2$. Find $f(-2)$ and $f(0)$.
18. $f(x) = x^4 - 4x^3 - 4x^2 + 24x - 9$. Find $f(3)$ and $f(9)$.
19. $f(x) = x^4 - 3x^3 - 13x^2 + 21x + 18$. Find $f(1)$ and $f(3)$.
20. $f(x) = 7x^4 + 37x^3 - x^2 - 14x + 4$. Find $f(1)$ and $f(-5)$.

13–3. THE REMAINDER THEOREM

In Section 13–2, it was shown that the remainder in the division of a polynomial by a binomial $x - r$ can be found without actually performing the division. Thus, in establishing (13–3), we have proved the following theorem.

Remainder Theorem. If a polynomial $f(x)$ is divided by $x - r$, the remainder is the value of $f(x)$ for $x = r$; that is, the remainder is $f(r)$.

It follows from this theorem that $f(x)$ is exactly divisible by $x - r$ if and only if $f(r) = 0$. Hence, we have proved also the following theorem.

Factor Theorem. If $f(r) = 0$, then $x - r$ is a factor of the polynomial $f(x)$, and conversely.

Example 13–3. Is $x + 3$ a factor of $x^4 - 2x^3 + 3x^2 - 5$?

Solution: Here $f(x) = x^4 - 2x^3 + 3x^2 - 5$. Also, $x - r = x + 3$, and $r = -3$. According to the factor theorem, $f(-3)$ must equal zero if $x + 3$ is to be a factor of $f(x)$. But $f(-3) = (-3)^4 - 2(-3)^3 + 3(-3)^2 - 5 = 81 + 54 + 27 - 5 = 157$. Therefore, since $f(-3) \neq 0$, $x + 3$ cannot be a factor of $x^4 - 2x^3 + 3x^2 - 5$.

Example 13–4. Given $f(x) = x^3 - 3x^2 + 5x - 6$. Show that $f(2) = 0$ and, therefore, that $x - 2$ is a factor of $f(x)$.

Solution: Since $f(x) = x^3 - 3x^2 + 5x - 6$, we have by substitution

$$f(2) = (2)^3 - 3(2)^2 + 5(2) - 6 = 0.$$

From the fact that $f(x)$ equals zero when $x = 2$, it follows from the factor theorem that $x - 2$ is a factor of $f(x)$. The student should check this result by synthetic division and find that

$$f(x) = x^3 - 3x^2 + 5x - 6 = (x - 2)(x^2 - x + 3).$$

Example 13–5. Find under what condition $(x + a)$ is a factor of $x^n + a^n$, where n is an integer and $a \neq 0$.

Solution: In this case, $f(x) = x^n + a^n$, and $f(-a) = (-a)^n + a^n$. This sum can equal zero only if $(-a)^n = -a^n$, that is, only when n is odd.

EXERCISE 13–2

In each of the problems from 1 to 18, determine if the second function is a factor of the first. If it is a factor, find another factor.

1. $2x^3 - 6x^2 + x + 6,\ x - 2.$

2. $x^3 + 4x^2 + 5x + 6,\ x^2 + x + 2.$

3. $x^3 + 2x^2 - 3x - 1,\ x - 1.$

4. $x^7 - 1,\ x - 1.$

5. $x^3 - x^2 - 11x + 15,\ x - 3.$

6. $x^5 + 243,\ x + 3.$

7. $x^8 - 256y^{16},\ x - 2y^2.$

8. $2x^3 + 3x^2 - 9x - 111,\ x - 1.$

9. $x^4 - 4x^3 - x^2 + 16x - 12,\ x + 1.$

10. $x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120,\ x + 1.$

11. $5x^3 + ab\,(5 - 6ab^2)x^2 - 2a^2b^2\,(5 + 3ab^2)x + 12a^4b^5,\ x - ab.$

12. $2x^4 - 3x^3 - 3x - 2,\ x + 2.$

13. $x^5 - 10x^4 + 18x^3 - 24x + 75,\ x - 2.$

14. $2x^4 - 31x^3 + 21x^2 - 17x + 10,\ x + 1.$

15. $12x^4 - 40x^3 - x^2 + 111x - 90,\ 2x - 3.$

16. $12x^3 - 22x^2 - 34x + 60,\ 3x + 5.$

17. $24x^4 - 122x^3 + 159x^2 - 7x - 60,\ 3x - 4.$

18. $24x^5 - 74x^4 - 85x^3 + 311x^2 - 74x - 120,\ 2x + 1.$

19. Show that the equation $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$ has the roots $1, 2, -1, -4$.

20. Find all roots of the equation $12x^4 - 40x^3 - x^2 + 111x - 90 = 0$, given that $3/2$ is a *double root,* that is, that $(x - 3/2)^2$ is a factor of the left side.

21. Prove that $a - b$ is a divisor of $a^n - b^n$ for every positive integral value of n.

22. Prove that $a + b$ is a divisor of $a^n - b^n$ if n is a positive even integer.

23. Prove that $a + b$ is a divisor of $a^n + b^n$ if n is a positive odd integer.

24. Prove that neither $a + b$ nor $a - b$ is a divisor of $a^n + b^n$ if n is a positive even integer.

13–4. THE FUNDAMENTAL THEOREM OF ALGEBRA

We usually assume that every algebraic equation with real or complex coefficients has at least one real or complex root. Although the existence of such a root is not to be taken for granted, a proof is beyond the scope of this book. Accordingly, we shall accept the following theorem as true.

Fundamental Theorem of Algebra. Let $f(x)$ be a polynomial of degree n with complex coefficients. Then the algebraic equation $f(x) = 0$ has at least one complex root.

The first complete and rigorous proof of the fundamental theorem was given by Gauss in the beginning of the nineteenth century. Since that time, many proofs have appeared, but most require knowledge of the theory of complex functions.

By a repeated application of the fundamental theorem, it can be shown that the number of roots of any polynomial equation with real or complex coefficients is equal to the degree of the polynomial. We shall state the following theorem and give its proof.

Theorem. If $f(x)$ is a polynomial of degree n, the equation $f(x) = 0$ has exactly n roots.

Proof. Let $f(x)$ be a polynomial of degree n with real or complex coefficients. By the fundamental theorem, there is a number r_1 such that $f(r_1) = 0$. Then, by the factor theorem,

$$f(x) = (x - r_1) \cdot q_1(x),$$

where $q_1(x)$ is a polynomial of degree $n - 1$ whose leading coefficient is a_0.

Likewise, $q_1(x) = 0$ has a root by the fundamental theorem. If we denote this root by r_2, then $q_1(r_2) = 0$. Also,

$$q_1(x) = (x - r_2) \cdot q_2(x),$$

where $q_2(x)$ is of degree $n - 2$ with leading coefficient a_0.

Similarly, $q_2(x) = 0$ also has a root. We can continue in this way until we come to a polynomial of the first degree with root r_n. Then

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n), \tag{13–5}$$

where a_0 is the coefficient of x in our final first-degree polynomial.

Since $f(x)$ equals zero when we substitute for x any one of the n numbers $r_1, r_2, \cdots, r_n$, it follows that the equation $f(x) = 0$ has at least the roots $r_1, r_2, \cdots, r_n$.

Moreover, there are no other roots. For, suppose that r is some root other than $r_1, \cdots, r_n$. Substitution of r for x in (13–5) yields

$$f(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n).$$

The right side of this equation cannot equal zero, since no one of the factors can equal zero. Therefore, $f(r) \neq 0$, and $f(x) = 0$ has no more than the n roots found before.

It may happen that a certain root appears more than once among the numbers $r_1, r_2, \cdots, r_n$. In that case it will be counted as many times as the corresponding equal factors appear in (13–5). If a certain factor $(x - r)$ appears m times in (13–5), then r is said to be a *root of multiplicity m*. A root is called a *simple root*, a *double root*, a *triple root*, and so on, the proper name depending on how many times the same factor appears. Hence, combining the statements that "$f(x) = 0$ has at least n roots" and "$f(x) = 0$ has no more than n roots," we can conclude that a polynomial equation of the nth degree has exactly n roots, a root of multiplicity m being counted as m roots.

It is to be noted that a rigorous proof of this theorem requires the use of induction. (See Chapter 16.)

13–5. PAIRS OF COMPLEX ROOTS OF AN EQUATION

We wish to remind the student that while an equation of the nth degree has n complex roots, the number of *real* roots may be less than the degree of the equation. For example, $x^2 + 1 = 0$ has no real roots. Determination of the number of real roots may be simplified by use of the following theorem.

Theorem. If all the coefficients of $f(x) = 0$ are real numbers, and if the complex number $a + bi$ is a root of $f(x) = 0$, then the conjugate $a - bi$ is also a root. It is understood that a and b are real and $b \neq 0$.

Proof. Let x in $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ be replaced by $a + bi$. Then we have

$$f(a + bi) = a_0(a + bi)^n + a_1(a + bi)^{n-1} + \cdots + a_n.$$

If we expand the powers of $a + bi$ by the binomial theorem and simplify the resulting expression, then all terms which contain even powers of i will be real, while all terms which contain odd powers of i will be pure imaginary. Denote by P the aggregate real part, and by Q the aggregate imaginary part. Then we have

$$f(a + bi) = P + Qi = 0.$$

Hence, in accordance with (11–4), $P = Q = 0$.

Now, replace x by $a - bi$ in $f(x) = 0$. In those terms of $f(a - bi)$ in which $-bi$ is raised to an even power, the result will remain the same as in $f(a + bi)$. However, all terms in $f(a - bi)$ in which $-bi$ is raised to an odd power will have their signs changed. Hence,

$$f(a - bi) = P - Qi.$$

But, since we have shown that $P = Q = 0$, we conclude that

$$P - Qi = 0.$$

In other words, $a - bi$ is also a root of $f(x) = 0$.

Example 13–6. Solve $x^4 - x^3 - 2x^2 + 6x - 4 = 0$, one root being $1 + i$.

Solution: According to the last theorem, both $1 + i$ and $1 - i$ are roots of the given equation. Using Eqs. (12–3) and (12–4) for the sum and product of two roots, we find that these conjugate complex numbers are roots of $x^2 - 2x + 2 = 0$. Dividing the original polynomial by this quadratic function, we get the quotient $x^2 + x - 2$. Solution of the equation $x^2 + x - 2 = 0$ yields the remaining desired roots, $x = -2$ and $x = 1$.

EXERCISE 13–3

1. Solve $x^3 + x^2 - 2x + 12 = 0$, one root being $1 + \sqrt{3}i$.
2. Solve $x^3 + 3x^2 + 12x - 16$, one root being $-2 - 2\sqrt{-3}$.
3. Solve $x^4 - 2x^3 - 7x^2 + 18x - 18 = 0$, one root being $1 - i$.
4. Solve $x^4 + 3x^3 + 7x^2 + 6x + 4 = 0$, one root being $-1 - \sqrt{3}i$.
5. Solve $x^5 - 8x^4 + 27x^3 - 46x^2 + 38x - 12 = 0$, one root being $2 - \sqrt{2}i$, and one root being 1.
6. Solve $x^4 + x^2 + 1 = 0$ by considering the equation to be a quadratic equation in x^2.
7. Find a real cubic equation, two of whose roots are 2 and $1 + 2i$.
8. Find a real equation of lowest degree having the roots i and $1 - i$.

13–6. THE GRAPH OF A POLYNOMIAL FOR LARGE VALUES OF x.

In graphing a polynomial function, it is helpful to know the location of points on the curve for numerically large values of x. It can be shown that, when x is numerically sufficiently large, the term $a_0 x^n$ of highest degree is numerically larger than the sum of all the other terms combined. Therefore, the sign of this term determines the sign of the entire polynomial.

Let us consider the values of the function $f(x) = x^3 + 5x^2 - 7x - 13$ as x assumes various values from left to right along the x-axis, that is, for increasing values of x. The results may be tabulated conveniently as follows:

x	x^3	$5x^2 - 7x - 13$	$x^3 + 5x^2 - 7x - 13$
-10	-1000	$500 + 70 - 13 = 557$	-443
-8	-512	$320 + 56 - 13 = 376$	-136
-6	-216	$180 + 42 - 13 = 209$	-7
0	0	$0 + 0 - 13 = -13$	-13
5	125	$125 - 35 - 13 = 77$	202
10	1000	$500 - 70 - 13 = 417$	1417

When x is negative, but numerically sufficiently large, it is seen that $f(x)$ is negative. Thus, when $x = -10$, $x^3 = -1000$, while the sum of other terms, or the value of $5x^2 - 7x - 13$, is only 557; and $f(-10) = -443$. For points far enough to the right of the origin, say for $x = 10$, $f(x)$ is positive. For example, $f(10) = 1000 + 417 = 1417$. Hence, the graph of $y = f(x)$ would be located below the x-axis on the left but would rise above the x-axis as x gets larger and larger.

As a second illustration, let us consider the function $f(x) = x^4 + 7x^2 - 8x + 10$. Here $f(x)$ is positive for large numerical values of x, regardless of the sign of x. Hence, in this case the graph

would be above the x-axis for large numerical values of x to both right and left of the origin.

The following helpful conclusion can be drawn from the preceding discussion involving large values of x.

When x is sufficiently large and positive, $f(x)$ has the same sign as the leading coefficient a_0. When x is negative and sufficiently large numerically, $f(x)$ has the same sign as a_0 when n is even, and has the opposite sign when n is odd.

The following symbols are sometimes found in discussions of the values of functions for numerically large values of x. When the symbolic statement $f(+\infty) > 0$ is used, what is meant is that, for all sufficiently large positive values of x, $f(x)$ is positive. Similarly, the statement $f(+\infty) < 0$ means that for all sufficiently large positive values of x, $f(x)$ is negative; the statement $f(-\infty) > 0$ means that, for all negative values of x which are sufficiently large numerically, $f(x)$ is positive; and $f(-\infty) < 0$ means that, for all negative values of x which are sufficiently large numerically, $f(x)$ is negative.

Thus, if $f(x) = x^3 + 5x^2 - 7x - 13$, $f(-\infty) < 0$ while $f(+\infty) > 0$. However, if $f(x) = x^4 + 7x^2 - 8x + 10$, $f(-\infty) > 0$ and $f(+\infty) > 0$.

13–7. ROOTS BETWEEN *a* AND *b* IF *f(a)* AND *f(b)* HAVE OPPOSITE SIGNS

Another helpful theorem relating to the roots of a polynomial equation is the following.

Theorem. If the coefficients of a polynomial $f(x)$ are real, and if a and b are real numbers such that $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one real root between a and b.

We shall not give a proof of this statement here, but shall merely mention the following geometric considerations. The graph of a polynomial is a continuous curve; that is, it has no "breaks." Therefore, if the points $(a, f(a))$ and $(b, f(b))$ lie on opposite sides of the x-axis, the graph apparently has to cross the x-axis at least once between these points.

13–8. RATIONAL ROOTS

The following theorem is fundamental for the solution of equations having integral coefficients.

Theorem. If the equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with integral coefficients has the rational root $\frac{c}{d}$, where c and d

are integers having no common factor > 1, then c is a divisor of the constant term a_n, and d is a divisor of the leading coefficient a_0.

Proof. We shall make use of a principle from the theory of numbers: If an integer c divides the product of two integers a and b and if b and c have no common divisor other than ± 1, then c is a divisor of a.

Let $\frac{c}{d}$ be a root of $f(x) = 0$, where c and d are integers with no common divisor other than ± 1. Then

$$a_0 \frac{c^n}{d^n} + a_1 \frac{c^{n-1}}{d^{n-1}} + \cdots + a_{n-1} \frac{c}{d} + a_n = 0.$$

Multiplication by d^n gives

$$a_0 c^n + a_1 c^{n-1} d + \cdots + a_{n-1} c d^{n-1} + a_n d^n = 0.$$

Since c divides all terms before the final one, c also divides that term. If now c is factored into primes, none of these primes is a divisor of d, and therefore of d^n. Thus each prime divides a_n, and so c itself divides a_n. In a similar fashion, it may be shown that d divides a_0.

Example 13–7. Find the rational roots of $3x^3 - 17x^2 + 15x + 7 = 0$.

Solution: The possible rational roots are of the form $\frac{c}{d}$, where c is a divisor of the constant 7 and d is a divisor of the coefficient 3. The only possible roots are

$$\pm 1, \pm 7, \pm 1/3, \pm 7/3.$$

Using synthetic division to check -1, we obtain the following result:

$$\begin{array}{rrrr|l} 3 & -17 & 15 & 7 & \underline{-1} \\ & -3 & 20 & -35 & \\ \hline 3 & -20 & 35 & -28 & \end{array}$$

By the remainder theorem, $f(-1) = -28$. Hence, -1 is not a root.

We see, however, that $f(0) = +7$. Therefore, by the theorem in Section 13–7, there must be a root between $x = -1$ and $x = 0$. Examining our list of possible roots, in the hope that a rational root lies between -1 and 0, we see that a possibility is $-1/3$. The check by synthetic division follows:

$$\begin{array}{rrrr|l} 3 & -17 & 15 & 7 & \underline{-1/3} \\ & -1 & 6 & -7 & \\ \hline 3 & -18 & 21 & 0 & \end{array}$$

Hence, $x = -1/3$ is a root. After dividing the given polynomial by $x + 1/3$, and equating the quotient $3x^2 - 18x + 21$ to 0, we obtain the quadratic $x^2 - 6x + 7 = 0$. This equation has the roots $3 \pm \sqrt{2}$, which are not rational numbers. Therefore, the only rational root is $x = -1/3$.

In this example, as is sometimes the case, it has been possible to find *all* the roots of the given equation.

EXERCISE 13–4

1. Show that $18x^3 - 33x^2 + 2x + 5 = 0$ has real roots between -1 and 0, between 0 and 1, and between 1 and 2. Find these three roots.
2. Find the rational roots of $2x^3 - 9x^2 + 3x + 4 = 0$.
3. Prove that $x^3 + 2x^2 - 3x - 5 = 0$ has at least one positive root.
4. Prove that $x^4 - x^3 + x^2 + x - 3 = 0$ has at least one positive root and at least one negative root.
5. Prove the following corollary of the theorem in Section 13–8. If $f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0$ has integral coefficients and has an integral root r, then r is a divisor of a_n.
6. Find the integral roots of $x^4 - 1 = 0$.
7. Show that the equation $x^2 + x + 1 = 0$ has no rational roots.
8. Solve the equation $x^3 - 1 = 0$.
9. Find all the integral roots of $x^3 + x^2 + x + 1 = 0$.
10. Show that 3 is a root of $x^3 - \frac{5}{2}x^2 - 2x + \frac{3}{2} = 0$. Why does this not contradict the theorem in Section 13–8 or that in Problem 5?

In each of the problems from 11 to 20, find all roots of the given equation.

11. $x^4 - 8x^2 + 16 = 0$.
12. $4x^3 - 16x^2 - 9x + 36 = 0$.
13. $3x^3 + x^2 + x - 2 = 0$.
14. $2x^3 + 3x^2 - 6x - 9 = 0$.
15. $2x^3 - x^2 + 2x - 1 = 0$.
16. $x^3 - 11x^2 + 37x - 35 = 0$.
17. $5x^3 - 13x^2 + 16x - 6 = 0$.
18. $x^4 - 8x^3 + 37x^2 - 50x = 0$.
19. $2x^3 - x^2 - 4x + 2 = 0$.
20. $3x^3 - 13x^2 + 13x - 3 = 0$.

14 Inequalities

14–1. INTRODUCTION

In previous chapters we have explained some methods of determining the roots of an equation. By applying these methods, one can find the values of an unknown for which a certain function of the unknown equals zero. Often, however, it is necessary to solve an inequality, that is, to discover for what values of the unknown a certain function is less than or greater than another function.

The present chapter is concerned primarily with the solution of inequalities, and the following discussion is essentially an extension of the study of the order relation undertaken in Section 1–8. We, therefore, recommend that the student thoroughly review Section 1–8 before starting the study of the present chapter. Since the solution of inequalities often involves the use of absolute values, a thorough mastery of Sections 1–9 and 1–10 is also a requirement.

The classification of inequalities corresponds to that of equalities or equations. As in the study of equations, there are two kinds of inequalities involving unknowns, namely, *absolute inequalities* and *conditional inequalities.*

An *absolute inequality* is an inequality that is satisfied by all values of the variable or variables for which the functions appearing are defined.

A *conditional inequality* is one that is true only for certain values of the variable or variables. Thus, $x^2 + 1 > 0$ (where x is real) is an absolute inequality, because it is true for every real value of x; but $x - 1 > 0$ is a conditional inequality, because it is valid only when $x > 1$.

14–2. PROPERTIES OF INEQUALITIES

The rules for dealing with inequalities are to some extent analogous to those for equations. In transforming inequalities, we shall have occasion to use the following elementary principles which

follow at once from the fundamental properties proved in Section 1–8.

Principle 1. If $a < b$, then $a \pm c < b \pm c$.
Here are three illustrations:
From $12 > 8$, it follows that $12 + 3 > 8 + 3$.
From $8 < 12$, it follows that $8 - 2 < 12 - 2$.
From $12 > 8$, it follows that $12 - 8 > 0$.

Principle 2. If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}\cdot$
Two illustrations are given here:
From $3 < 5$, it follows that $3 \cdot 2 < 5 \cdot 2$.
From $8 < 10$, it follows that $\frac{8}{2} < \frac{10}{2}\cdot$

Principle 3. If $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}\cdot$
Here is an illustration:
From $3 < 4$, it follows that $-3 > -4$.

14–3. SOLUTION OF CONDITIONAL INEQUALITIES

The process of *solution* of a conditional inequality consists in finding all values of the variable which satisfy the inequality. A solution consists of a *set of values* of the variable, rather than one or more isolated values as is usual in the case of a conditional equation.

The discussion in this section is limited to inequalities involving rational functions in only one variable. If this variable is x, the inequality can be written in the form $f(x) > 0$ or $f(x) < 0$. For instance, suppose we want to solve the inequality

$$x^2 - x > 2.$$

We may then obtain the following equivalent inequality:

$$f(x) = x^2 - x - 2 > 0.$$

It is easily seen that this transformed inequality has the same solution set as the original inequality.

In solving this transformed inequality, we find first the values of x, if there are any, for which $f(x)$ changes sign as x increases in magnitude. If $f(x)$ is a polynomial, such a change of sign occurs when $f(x) = 0$. In the example under consideration, changes of sign are obtained only at points where

$$f(x) = x^2 - x - 2 = 0.$$

Hence, we must find the roots of $x^2 - x - 2 = 0$. These are -1 and 2, and they determine on the x-axis three intervals throughout each of which $f(x)$ retains the same sign. In other words, to find the

solution of $f(x) > 0$, we find the interval or intervals within which $f(x)$ has the sign indicated in the given inequality. In the example under consideration this sign is positive, because $f(x)$ is to be > 0. The method is applied in Example 14–1.

In general, the solution of an inequality is obtained by equating the function $f(x)$ to zero and solving the resulting equation. If the inequality is of the form $\frac{p(x)}{q(x)} \gtrless 0$, where $p(x)$ and $q(x)$ are polynomials, it may be cleared of fractions by multiplication by $[q(x)]^2$. Since the square of any non-zero real number is positive, the sense of the inequality is not changed by multiplication by this factor. This leads to the form $f(x) \gtrless 0$, where $f(x)$ is a polynomial. The values of x for which $f(x)$ changes sign are called *critical values*. When the critical values are arranged in increasing order, they determine on the x-axis intervals, throughout each of which $f(x)$ cannot change sign. Consequently, the required solution is represented by the set of values of x for which $f(x)$ has the same sign as that indicated in the given inequality.

Note. In general, it can be shown that if a factor of $f(x)$ appears to an odd power (that is, if $f(x) = 0$ has roots of odd multiplicity), the function will change sign at values of x for which this factor vanishes. If a factor of $f(x)$ appears to an even power (that is, if $f(x) = 0$ has roots of even multiplicity), the function will not change sign at values of x for which this factor vanishes. Therefore, it is sufficient to test only one value of x in one interval. This test gives the sign of $f(x)$ in that interval. The sign of $f(x)$ in each of the other intervals can be quickly and easily determined from the multiplicity of the critical values. Substituting into $f(x)$ a value of x in each interval then provides a check of the solution.

Example 14–1. Solve the inequality

$$x^2 - x > 2.$$

Solution: An equivalent inequality is

$$f(x) = x^2 - x - 2 > 0.$$

From the preceding discussion it follows that the critical values are the roots of

$$x^2 - x - 2 = 0.$$

These roots are -1 and 2.

As shown in Fig. 14–1, the points -1 and 2 determine on the x-axis the following three intervals

(a) $x < -1$; (b) $-1 < x < 2$; (c) $x > 2$.

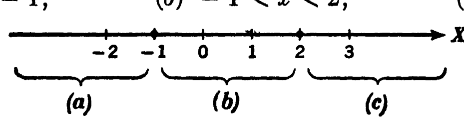

FIG. 14–1.

Throughout each of these intervals, $f(x)$ retains the same sign. This condition may also be seen from the graph of $y = x^2 - x - 2$ shown in Fig. 14-2. Here we note that in each of these intervals the curve lies either entirely above the x-axis or entirely below it.

The solution of $f(x) > 0$ can now be found by examining the sign of $f(x)$ in each of these intervals. Thus, for a value such as $x = -2$ in the interval (a), $f(-2) = (-2)^2 - (-2) - 2 = 4$. Hence, $f(x)$ is positive throughout the interval (a). In the graph of $y = x^2 - x - 2$ shown in Fig. 14-2, the curve lies above the x-axis to the left of $x = -1$.

For the value $x = 0$ in the interval (b), we have $f(0) = -2$. Hence, $f(x)$ is negative throughout this interval, and the graph lies below the x-axis.

For the value $x = 3$ in the interval (c), $f(3) = (3)^2 - (3) - 2 = 4$. So $f(x)$ is positive, and the graph again lies above the x-axis.

These same results may also be obtained by the following much shorter procedure: Select a value of x in the interval (b) for which $f(x)$ is easily evaluated. Such a value is $x = 0$. Since $f(0) = -2$, $f(x) < 0$ throughout this interval. Therefore, $f(x) > 0$ for intervals (a) and (c), because the sign of $f(x)$ changes at the critical values $x = -1$ and $x = 2$.

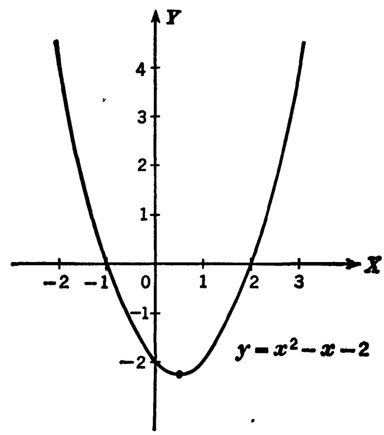

FIG. 14-2.

We see, therefore, that in the intervals (a) and (c), $f(x)$ has the sign indicated by the given inequality. Since $f(x)$ must be greater than zero, the solution set of the given inequality is described by $x < -1$ and $x > 2$.

Example 14-2. Determine the values of x for which $\sqrt{x^3 - 2x^2 - 3x}$ is real.

Solution: We shall solve the equivalent problem

$$f(x) = x^3 - 2x^2 - 3x \geqq 0.$$

Solving the equation $x^3 - 2x^2 - 3x = 0$, we find that the critical values are $-1, 0, 3$.

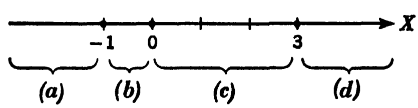

FIG. 14-3.

As shown in Fig. 14-3, these critical values determine on the x-axis the following four intervals:

(a) $x < -1$; (b) $-1 < x < 0$;
(c) $0 < x < 3$; (d) $x > 3$.

Throughout each interval $f(x)$ has the same sign.

In this example we may select $x = 1$ in the interval (c). Since $f(1) = (1)^3 - 2(1)^2 - 3(1) = -4$, $f(x) < 0$ throughout this interval.

As we proceed into the interval (b), we find that $f(x)$ changes sign when $x = 0$. Therefore, $f(x) > 0$ in the interval (b). Again $f(x)$ changes sign when $x = -1$ and becomes < 0 in the interval (a). Proceeding to the right from the interval (c) into the interval (d), we find that $f(x)$ changes sign when $x = 3$ and is > 0 in the interval (d).

Thus, we have the following results:

in (a), $f(x) < 0$; in (b), $f(x) > 0$;
in (c), $f(x) < 0$; in (d), $f(x) > 0$.

Hence, the inequality $f(x) > 0$ is satisfied in intervals (b) and (d). And, since the condition $x^3 - 2x^2 - 3x = 0$ is also allowed in the original problem, the values -1, 0, 3 are included in the solution. Therefore, the solutions for $f(x) = x^3 - 2x^2 - 3x \geqq 0$ are $-1 \leqq x \leqq 0$ and $x \geqq 3$. These are the values of x for which the original expression $\sqrt{x^3 - 2x^2 - 3x}$ is real.

Example 14–3. What values of x satisfy $\dfrac{x^3 - 3x^2}{x - 2} > 0$?

Solution: Clear the given inequality of fractions by multiplying by $(x - 2)^2$ and obtain $f(x) = (x - 2)(x^3 - 3x^2) > 0$. Solving the equation $x^2(x - 2)(x - 3) = 0$, we find that the critical values are 0, 0, 2, 3.

Hence, we have the following four intervals, as shown in Fig. 14–4:

(a) $x < 0$; (b) $0 < x < 2$; (c) $2 < x < 3$; (d) $x > 3$.

Throughout each of these intervals $f(x)$ has the same sign.

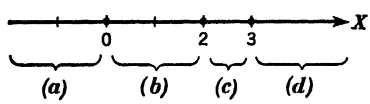

FIG. 14–4.

Let us initially test $f(x)$ for $x = 1$ in the interval (b). Since $f(1) = (1 - 2)[(1)^3 - 3(1)^2] = (-1)(-2) = 2$, it follows that $f(x) > 0$ throughout this interval.

We see that $f(x)$ does not change sign for the critical value $x = 0$, because $x = 0$ is a double root of $f(x) = 0$. Hence, $f(x) > 0$ in the interval (a).

As we proceed to the right from the interval (b) the function $f(x)$ changes sign for each of the critical values $x = 2$ and $x = 3$. Hence, $f(x) < 0$ in the interval (c), and $f(x) > 0$ in the interval (d).

Therefore, $f(x)$ is positive in the intervals (a), (b), and (d). That is, the solution set of the original inequality is described by $x < 2$, excluding the value $x = 0$, and $x > 3$.

Example 14–4. Solve the inequality $x^2 - 2x + 3 > 0$.

Solution: Let $f(x) = x^2 - 2x + 3$. Since the roots of $f(x) = 0$ are imaginary, there are no critical values. Hence, the graph of $y = f(x)$ lies either entirely above the x-axis or entirely below that axis.

Testing for $x = 0$, we find that $f(0) = 3$. This result indicates that the graph lies above the x-axis. Consequently, the inequality is satisfied for all real values of x.

Example 14–5. Solve the inequality $\left|\frac{2x-1}{3}\right| < 1.$

Solution: By Section 1–10, the inequality $|x - b| < a$ is equivalent to $b - a < x < b + a$. Hence, $\left|\frac{2x-1}{3}\right| < 1$, which is equivalent to $|2x - 1| < 3$, may be written as follows:

$$1 - 3 < 2x < 1 + 3 \quad \text{or} \quad -2 < 2x < 4.$$

We thus find that the solution set of the original inequality is described by $-1 < x < 2$.

Alternate Solution: We may proceed by solving individually the two inequalities $-1 < \frac{2x-1}{3}$ and $\frac{2x-1}{3} < 1$ and determining the common solutions. For $-1 < \frac{2x-1}{3}$, we have $-3 < 2x - 1$, or $-1 < x$. For $\frac{2x-1}{3} < 1$, we have $2x - 1 < 3$, or $x < 2$. The common solutions satisfy the inequalities $-1 < x$ and $x < 2$. So we again have $-1 < x < 2$.

EXERCISE 14–1

In each of the following problems, solve the given conditional inequality or inequalities.

1. $x - 3 < 0.$ **2.** $x + 1 < 0.$ **3.** $x + 5 > 0.$

4. $x - 1 \geqq 0.$ **5.** $4x - 16 < 0.$ **6.** $4x - 16 > 0.$

7. $6x + 3 < 0.$ **8.** $4x - 8 > 3x - 10.$ **9.** $4x < -3x - 7.$

10. $3 - 4x < 2x + 1.$ **11.** $-3 < 6x < 3.$ **12.** $|x - 1| < 3.$

13. $0 < \frac{x+1}{2} + \frac{1}{3} < 1.$ **14.** $-1 < \frac{x-3}{2} + \frac{3}{5} < 1.$ **15.** $|2x - 3| < 4.$

16. $\left|\frac{x}{2} - \frac{3}{5}\right| < 2.$ **17.** $x^2 < 169.$ **18.** $x^2 > 144.$

19. $2x^2 \leqq 32.$ **20.** $\frac{x^2}{2} - \frac{3}{4} < 0.$ **21.** $x(3x + 2) < 1.$

22. $4x^2 + 5x < -1.$ **23.** $\frac{10}{x^2} < 3 - \frac{1}{x}.$ **24.** $\frac{11}{x^2} < -\frac{6}{x} - 1.$

25. $3x^2 - 3x < -4.$ **26.** $3x^2 + 6x < 9.$

27. $(x + 1)(x + 2)(x + 3) < 0.$ **28.** $(x - 4)(x + 5)(x - 6) > 0.$

29. $(x - 1)(x - 2)(x - 3) > 0.$ **30.** $(2x - 1)(x + 2)(3x + 1) \leqq 0.$

31. $\sqrt{x^2 - 25}$ is real. **32.** $\sqrt{x^2 - 5x + 6}$ is real.

33. $\frac{x(x-1)^2}{x+2} > 0.$ **34.** $\sqrt{\frac{x-1}{x+1}} < 1.$

35. $\left|\frac{x+1}{x}\right| < 1.$ **36.** $\frac{x}{x^2 - 2x - 3} < 0.$

14–4. ABSOLUTE INEQUALITIES

To prove the truth of an absolute inequality, one must use the known properties of the order relation. When none of these seems readily applicable to the given inequality, it may be helpful to replace this inequality by an equivalent one which may be more easily treated. Repeated replacements may have to be made.

In carrying out a sequence of replacements, one need not verify equivalence at each stage, provided that the final inequality can be shown to imply the original one.

The methods for proving absolute inequalities may be used also to prove theorems involving inequalities, as in Example 14–7.

Example 14–6. Prove that $a^2 + b^2 \geqq 2ab$ for all real numbers a and b.

Solution: The given inequality is equivalent, by Principle 1, Section 14–2, to

$$a^2 - 2ab + b^2 \geqq 0,$$

that is, to

$$(a - b)^2 \geqq 0.$$

This last inequality is true, because the square of every real number is nonnegative. Therefore, the original inequality is true also.

Example 14–7. If a, b, c, and d are distinct positive real numbers, and if $\frac{a}{b} < \frac{c}{d}$, prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Solution: The inequality $\frac{a}{b} < \frac{a+c}{b+d}$ is equivalent, by Principle 2, Section 14–2, to

$$ab + ad < ab + bc.$$

This inequality is equivalent, by Principle 1, to

$$ad < bc.$$

This inequality is true because it is equivalent to the given condition $\frac{a}{b} < \frac{c}{d}$ by Principle 2. Therefore, the inequality $\frac{a}{b} < \frac{a+c}{b+d}$ is true also.

Similarly, the inequality $\frac{a+c}{b+d} < \frac{c}{d}$ is equivalent, by Principle 2, to

$$ad + cd < bc + cd,$$

and this inequality, by Principle 1, is equivalent to

$$ad < bc.$$

This last inequality again is equivalent to the given condition $\frac{a}{b} < \frac{c}{d}$. Hence, the inequality $\frac{a+c}{b+d} < \frac{c}{d}$ is true also. Therefore, it follows that the original inequalities $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ are true.

EXERCISE 14–2

1. Prove that $a^2 + 1 \geqq 2a$ if a is real. Note that $a^2 + 1 = 2a$ only when $a = 1$.

2. Prove that $\dfrac{a+b}{2} \geqq \dfrac{2ab}{a+b}$ if $a > 0$ and $b > 0$.

3. Prove that $a^2 > a$ if $a > 1$; and that $a^2 < a$ if $0 < a < 1$.

4. Prove that $a > a^3$ if $0 < a < 1$.

5. Prove that $\dfrac{a}{b} + \dfrac{b}{a} > 2$ if a and b are positive and $a \neq b$.

6. Prove that $a^2 + b^2 + c^2 \geqq ab + bc + ca$. (Hint: From Example 14–6, $a^2 + b^2 \geqq 2ab$, $b^2 + c^2 \geqq 2bc$, and $c^2 + a^2 \geqq 2ca$. Add these inequalities.)

7. If a and b are two positive real numbers, the quantities $\dfrac{a+b}{2}$, $\sqrt{ab}$, and $\dfrac{2ab}{a+b}$ are called, respectively, the *arithmetic mean* (A), the *geometric mean* (G), and the *harmonic mean* (H) of a and b. Prove that $H < G < A$, except when $a = b$. (In this case, we have $A = G = H$.)

8. Prove that $a^3 + b^3 > 3ab(a - b)$, if a and b are positive and $a > b$.

9. Prove that $ab + cd \leqq 1$, if a, b, c, and d are positive, and if $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$.

10. Prove that $\dfrac{a+c}{b+c} \gtrless \dfrac{a}{b}$, according as $a \lessgtr b$, if a, b, and c are positive.

11. Prove that $a^3b + ab^3 < a^4 + b^4$, if $a \neq b$.

12. If $z = x + yi$ is a complex number, the modulus or absolute value of z is denoted by $|z| = \sqrt{x^2 + y^2}$. Show that $|z_1 + z_2| \leqq |z_1| + |z_2|$ and that $|z_1 - z_2| \geqq |z_1| - |z_2|$, for all complex numbers z_1 and z_2.

13. If z_1, z_2, and z_3 are complex numbers, prove that $|z_1 + z_2| \leqq |z_1 + z_3| + |z_2 - z_3|$.

14. If z is a complex number, prove that $0 \leqq \big||z| + z\big| \leq 2|z|$.

15 Progressions

15–1. SEQUENCES AND SERIES

Sequences. An *infinite sequence,* called more simply a *sequence* or sometimes a *progression,* is a single-valued function whose domain of definition is the set of positive integers. A *finite sequence* is a single-valued function whose domain consists of the integers $1, 2, \cdots, m$ for some positive integer m.

In specifying a sequence, it is necessary to give a definite rule of correspondence which assigns to each integer n a single definite number, or *term,* of the sequence. This term may be denoted by a_n. In particular, there is a first term a_1 corresponding to the integer 1, a second term a_2 corresponding to the integer 2, and so on. The sequence may be specified by the array of numbers

$$a_1, a_2, a_3, \cdots, a_n, \cdots. \tag{15–1}$$

This sequence is denoted briefly by $\{a_n\}$. (As usual, throughout this discussion, a row of dots $\cdots$ stands for numbers assumed to be present but not written.) A finite sequence has a "last" term and may be designated by $a_1, a_2 \cdots, a_m$ or simply by $\{a_n\}$.

The nth term, or general term, of a sequence is denoted by a_n. From the rule specifying the nth term for each n, we obtain the first, second, third, and other terms of the sequence by substituting for n the values 1, 2, 3, and so on in turn. For example, if $a_n = 1/n$, the sequence is

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots.$$

We should note that $\{a_n\}$ is the symbol for the sequence or function as a whole, whereas a_n is the symbol for the nth term or value of the function corresponding to the integer n.

There are many methods for specifying the function in the definition of a sequence. Two of these methods follow:

Explicit Formula. In one method, the nth term is given in terms of n itself by means of an explicit formula.

Here are a few illustrations.

If $a_n = n$, the sequence is $1, 2, 3, \cdots$.

If $a_n = \dfrac{n}{n^2+1}$, the sequence is $\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}, \cdots$.

If $a_n = \dfrac{n}{2n-1}$, the sequence is $1, \dfrac{2}{3}, \dfrac{3}{5}, \cdots$.

Recursion Rule. In another method, one or more of the first several values of a_n are given explicitly, and a rule is then given whereby a_n can be calculated from some or all of its predecessors.

A few illustrations are given here.

Let $a_{n+1} = a_n{}^2 + 1$ with $a_1 = 0$. Then

$$\begin{aligned} a_2 &= a_1{}^2 + 1 = 0^2 + 1 = 1, \\ a_3 &= a_2{}^2 + 1 = 1^2 + 1 = 2, \\ a_4 &= a_3{}^2 + 1 = 2^2 + 1 = 5, \\ &\cdots\cdots\cdots\cdots. \end{aligned}$$

Let $a_{n+1} = (n+1)a_n$ with $a_1 = 1$. Then

$$\begin{aligned} a_2 &= 2a_1 = 2 \cdot 1 = 2, \\ a_3 &= 3a_2 = 3 \cdot 2 = 6, \\ a_4 &= 4a_3 = 4 \cdot 6 = 24, \\ &\cdots\cdots\cdots\cdots. \end{aligned}$$

Note that in this example $a_n = n!$

Let $a_{n+2} = a_{n+1} + a_n$ with $a_1 = 0$ and $a_2 = 1$. Then

$$\begin{aligned} a_3 &= a_2 + a_1 = 1 + 0 = 1, \\ a_4 &= a_3 + a_2 = 1 + 1 = 2, \\ a_5 &= a_4 + a_3 = 2 + 1 = 3, \\ &\cdots\cdots\cdots\cdots. \end{aligned}$$

Series. Let $\{a_n\}$ be a given sequence of terms $a_1, a_2, \cdots, a_n, \cdots$. Form a new sequence $\{s_n\}$, where s_n is obtained by adding the first n terms of $\{a_n\}$. The sequence of *partial sums* is then given by

$$\begin{aligned} &s_1 = a_1, \quad s_2 = a_1 + a_2, \cdots, \\ &s_n = a_1 + a_2 + \cdots + a_n, \cdots \qquad (n = 1, 2, 3, \cdots). \end{aligned}$$

The sequence $\{s_n\}$ formed in this way is called the *(infinite) series* based on the given sequence $\{a_n\}$.

The series as just defined is usually written in the following abbreviated form:

$$a_1 + a_2 + \cdots + a_n + \cdots. \tag{15–2}$$

Two illustrations of series follow.

Illustration 1. Let $a_n = \frac{1}{n}$ and $s_n = a_1 + a_2 + \cdots + a_n$. Then the partial sums are given by

$$s_1 = a_1 = 1,$$
$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2},$$
$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

The series may then be written

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Illustration 2. Let $a_n = \frac{1}{n^2 + n}$ and $s_n = a_1 + a_2 + \cdots + a_n$. Then the partial sums are given by

$$s_1 = a_1 = \frac{1}{2},$$
$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$
$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

The series is

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n^2 + n} + \cdots.$$

Limit of a Sequence. One of the most important questions relating to sequences is whether or not a given sequence $\{a_n\}$ has a limit as n increases indefinitely. If such a limit $\mathcal{L}$ exists, the sequence is said to be *convergent* or to *converge* to the limit $\mathcal{L}$. Symbolically this statement may be expressed as follows:

$$\lim_{n \to \infty} a_n = \mathcal{L}.$$

This notation is read, "the limit of a_n as n increases indefinitely is $\mathcal{L}$."

In order to help make the concept of limit clear, we shall consider the sequence given by $a_n = \frac{1}{n} \cdot$ The terms are

$$a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, \ldots, a_n = \frac{1}{n}, \ldots.$$

When we examine these terms, we notice that the larger the number n is, the smaller the term becomes. In other words, as n increases, the closer to zero is the number $\frac{1}{n} \cdot$ In fact, $\frac{1}{n}$ can be made as small as we please, if we merely choose n sufficiently large. For example,

$\frac{1}{n} < \frac{1}{100}$ for every n larger than 100. We see also that $\frac{1}{n} < \frac{1}{1000}$ for every n larger than 1000. We may conclude that for an arbitrary real positive number d, we can find a value of n, say any integer $N \geqq \frac{1}{d}$, such that for all integers $n > N$, it is true that $\frac{1}{n} < d$. The limit 0 and the sequence $\left\{\frac{1}{n}\right\}$ are therefore related as indicated by the following statement.

The sequence $\left\{\frac{1}{n}\right\}$ *converges to the limit* $\mathcal{L} = 0$, if to each arbitrary positive number d there corresponds an integer $N > 0$ such that $0 - d < \frac{1}{n} < 0 + d$ for every $n > N$.

In general, the limit $\mathcal{L}$ of a sequence may be defined as follows:

Definition of Limit of Sequence. A sequence $\{a_n\}$ *converges to the limit* $\mathcal{L}$, if to each arbitrary real number $d > 0$, there corresponds a positive integer N such that $\mathcal{L} - d < a_n < \mathcal{L} + d$ for every $n > N$. This definition may also be put as follows:

Definition. A sequence $\{a_n\}$ *converges to the limit* $\mathcal{L}$, if for each number $d > 0$ there exists a positive integer N such that

$$|\mathcal{L} - a_n| < d \text{ for every } n > N.$$

Convergence of a Series. We shall again consider the infinite series

$$a_1 + a_2 + \cdots + a_n + \cdots, \tag{15–2}$$

which is the sequence of partial sums

$$s_1, s_2, \cdots, s_n, \cdots$$

of the sequence $\{a_n\}$. By the following definition the series is convergent if the sequence of partial sums is convergent.

Definition. If the sequence of partial sums of the infinite series (15–2) converges to a limit, and if $\lim_{n \to \infty} S_n = S$, then the series is said to *converge to the limit* S, and S is called the *sum* of the infinite series.

The new use of the word "sum" for the value S of an infinite series is perhaps unfortunate, for it seems meaningless to talk about adding up the terms of an infinite series. Actually, S is not a sum, but it is rather the limit of a sequence of partial sums of the series.

If a series does not converge to a limit as n becomes infinite, we say that it is *divergent*, or that it *diverges*.

EXERCISE 15–1

1. Given $a_1 = 1$ and $a_{n+1} = n + a_n$. Find the five terms of the sequence $\{a_n\}$.
2. Given $a_1 = 4$, $a_2 = 3$, and $a_{n+2} = 2a_{n+1} - a_n$. Find the first six terms of the sequence $\{a_n\}$.
3. Given $a_1 = 1$, $a_2 = 2$, and $a_{n+2} = (n + 1)a_{n+1} - na_n$. Find the first six terms of the sequence $\{a_n\}$.
4. Show that the sequence $a_n = 3^n$ satisfies the recursion rule $a_{n+2} = a_{n+1} + 6a_n$.
5. Show that the sequence $a_n = (-2)^n$ also satisfies the recursion relationship of Problem 4.
6. Assuming that A and B are any real numbers whatsoever, show that the sequence $a_n = A3^n + B(-2)^n$ satisfies the relationship of Problem 4.
7. Given $a_{n+1} = 2a_n + 1$ with $a_1 = 2$. Let $s_n = a_1 + a_2 + \cdots + a_n$. Find $a_1, a_2, \cdots, a_5$ and $s_1, s_2, \cdots, s_5$.
8. Show that if $s_n = a_1 + a_2 + \cdots + a_n$, then $s_{n+1} - s_n = a_{n+1}$.
9. Prove that $a_n = s_n - s_{n-1}$ for $n > 1$, given $a_1 = s_1$.
10. If $s_n = n^2$, show that $a_n = 2n - 1$ and therefore that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

15–2. ARITHMETIC PROGRESSIONS

An *arithmetic progression* is a sequence of numbers in which each term after the first is obtained from the preceding one by adding to it a fixed number; this number is called the *common difference.*

Note that the common difference may be found by subtracting any term of the sequence from the one that follows. Thus, 1, 5/2, 4, 11/2, is an arithmetic progression with the common difference 3/2, since $11/2 - 4 = 4 - 5/2 = 5/2 - 1 = 3/2$. Also, 5, 1, −3, −7 is an arithmetic progression with the common difference −4, since $-7-(-3) = -3 -1 = 1 - 5 = -4$.

It follows, therefore, that a necessary and sufficient condition that three numbers A, B, and C form an arithmetic progression is $C - B = B - A$.

15–3. THE GENERAL TERM OF AN ARITHMETIC PROGRESSION

Let a denote the first term of an arithmetic progression, and let d denote the common difference. Then, by definition, an arithmetic progression with n terms may be written as follows:

$$a, a + d, a + 2d, a + 3d, \cdots, a + (n - 1)d.$$

Hence if l_n represents the value of the nth term,

$$l_n = a + (n - 1)d. \tag{15–3}$$

Also, we may write an arithmetic progression of n terms in the following manner:

$$a,\ a + d,\ a + 2d, \cdots, l_n - 2d,\ l_n - d,\ l_n.$$

We shall be concerned with five quantities in connection with an arithmetic progression. These are the first term a, the number of terms n, the nth term l_n, the difference d, and the sum S_n of the n terms.

15–4. SUM OF THE FIRST n TERMS OF AN ARITHMETIC PROGRESSION

Let S_n represent the sum of the first n terms of an arithmetic progression. If we write the indicated sum in both direct and reverse orders, we have

$$S_n = a + (a + d) + (a + 2d) + \cdots + (l_n - 2d) + (l_n - d) + l_n,$$

and

$$S_n = l_n + (l_n - d) + (l_n - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

Adding the right sides, we have n terms each of which is $a + l_n$. Thus,

$$2S_n = (a + l_n) + (a + l_n) + \cdots + (a + l_n) + (a + l_n) = n(a + l_n).$$

Therefore,

$$S_n = \frac{n}{2}(a + l_n). \tag{15–4}$$

If we substitute $a + (n - 1)d$ from (15–3) for l_n in (15–4), we have another useful form for the sum. This is

$$S_n = \frac{n}{2}[2a + (n - 1)d]. \tag{15–5}$$

Example 15–1. Determine which of the following sequences are arithmetic progressions:

a) 3, 7, 10; *b*) 6, 1, -4; *c*) $3x - y$, $4x + y$, $5x + 3y$.

Solution: *a*) Since the differences $10 - 7$ and $7 - 3$ are not equal, the sequence 3, 7, 10 is not an arithmetic progression.

b) In the second sequence, $-4 - 1 = 1 - 6 = -5$. Since these differences are equal, the sequence 6, 1, -4 is an arithmetic progression.

c) We find that $(5x + 3y) - (4x + y) = x + 2y$ and $(4x + y) - (3x - y) = x + 2y$. Since there is a common difference, the given sequence is an arithmetic progression.

Example 15–2. Find the twelfth term, and also the sum of the first 12 terms, of the arithmetic progression 4, 7, 10, $\cdots$.

Solution: We have $a = 4$, $n = 12$, and $d = 3$. Then, by (15–3),

$$l_{12} = a + (n - 1)d = 4 + (12 - 1)3 = 37.$$

Also, by (15–4),

$$S_{12} = \frac{n}{2}(a + l_n) = \frac{12}{2}(4 + 37) = 246.$$

Example 15–3. The third term of an arithmetic progression is 3/4, and the sixth term is 3/2. Find the twenty-second term.

Solution: By (15–3), $l_3 = a + 2d$ and $l_6 = a + 5d$. Thus, we have

$$\begin{cases} a + 2d = 3/4, \\ a + 5d = 3/2. \end{cases}$$

By solving these two linear equations, we find that $a = 1/4$ and $d = 1/4$. By (15–3), $l_{22} = 1/4 + 21(1/4) = 11/2$.

Example 15–4. Find each value of x for which the three quantities $3x - 5$, $x + 4$, $3x - 2$ form an arithmetic progression.

Solution: Applying the condition $C - B = B - A$, we have

$$(3x - 2) - (x + 4) = (x + 4) - (3x - 5).$$

Solving, we obtain $x = \frac{15}{4}\cdot$ The arithmetic progression is $\frac{25}{4}, \frac{31}{4}, \frac{37}{4}\cdot$

15–5. ARITHMETIC MEANS

The terms of an arithmetic progression between any two given terms are called *arithmetic means* between the given terms.

If we let the given terms be a and l_n, any number, say k, of means may be inserted between a and l_n by using the formula $l_n = a + (n - 1)d$ with $n = k + 2$. As soon as we have found d, we can insert the required means.

Example 15–5. Insert three arithmetic means between 0 and 1.

Solution: Let $a = 0$, $l_n = 1$, and $n = 3 + 2 = 5$. Then $1 = 0 + 4d$, and $d = \frac{1}{4}\cdot$ Hence, the three means are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\cdot$

If only a single arithmetic mean is to be inserted between two given numbers, then the inserted value is called *the arithmetic mean* of the given numbers. Thus, if a, x, b form an arithmetic progression, x is called the *arithmetic mean* of a and b. Since $b - x = x - a$,

$$x = \frac{a + b}{2}\cdot$$

Thus, the single arithmetic mean of two numbers is equal to one-half their sum.

EXERCISE 15–2

In each of the problems from 1 to 9, determine if the given sequence is an arithmetic progression. Find the next two terms of the extension of each arithmetic progression.

1. 5, 8, 11, 14.
2. $-1, 7, 13, 19.$
3. $-10, -3, 4, 11.$
4. 4, 12, 19, 27.
5. $-\frac{1}{4}, \frac{1}{2}, 1, \frac{7}{4}.$
6. $\frac{3}{8}, \frac{1}{16}, -\frac{1}{4}, -\frac{5}{16}.$
7. $a, b, 2b - a, 3b - 2a.$
8. $a + b, a - b, a - 2b, a - 3b.$
9. $\frac{a+b}{2}, a, \frac{3a-b}{2}.$

In each of the problems from 10 to 19, find l_n and S_n for the arithmetic progression.

10. $2, 8, 14, \cdots$ to 12 terms.
11. $3, 6, 9, \cdots$ to 26 terms.
12. $22, 18, 14, \cdots$ to 7 terms.
13. $1, 2, 3, \cdots$ to 10 terms.
14. $2, 4, 6, \cdots$ to 50 terms.
15. $1, 3, 5, \cdots$ to 75 terms.
16. $a, 2a, 3a, \cdots$ to 10 terms.
17. $0.2, 0.5, 0.8, \cdots$ to 20 terms.
18. $1, 8, 15, \cdots$ to 35 terms.
19. $1, 2, 3, \cdots$ to 100 terms.

In each of the problems from 20 to 27, three quantities relating to an arithmetic progression are given. Find the other two quantities.

20. $a = 5, l_4 = 36, n = 4.$
21. $a = 10, n = 10, d = 10.$
22. $S_{21} = 653, n = 21, a = 6.$
23. $n = 45, d = \frac{1}{2}, S_{45} = 63.$
24. $l_{21} = 8, n = 21, d = \frac{1}{5}.$
25. $l_n = 7, S_n = 52, d = \frac{1}{2}.$
26. $a = -\frac{1}{2}, d = \frac{3}{2}, S_n = \frac{125}{2}.$
27. $a = 1, S_n = 45, d = 1.$
28. Find the sum of the first 100 even integers.
29. Find the sum of the first n odd integers.
30. Insert five arithmetic means between 20 and 30.
31. Insert ten arithmetic means between 100 and 40.
32. Insert six arithmetic means between -3 and -2.
33. Find the arithmetic mean of 10 and 56 and that of 4 and 28.
34. Find the arithmetic mean of 28 and 65 and that of 33 and 78.
35. Insert k arithmetic means between $\frac{1}{a}$ and a, where $a \neq 0$.
36. A display of cans in a grocery store is in the form of a pyramid whose base is an equilateral triangle. If each side of the base contains 20 cans and the number of cans decreases by one for each successive row, how many cans are in the display?
37. A lottery contains tickets numbered consecutively from 1 to 100. Customers draw tickets and pay according to the number of the ticket, except that tickets numbered above 50 cost just 50 cents each. How much money is collected if all tickets are sold?
38. Determine x so that $x, x - 2, 3x$ will be an arithmetic progression.
39. The sum of the first and fourth terms of an arithmetic progression is 20. The sum of the third and twelfth terms is 36. Find the sum of the first 15 terms.
40. Find the sum of all multiples of 5 from 100 to 1,000, inclusive.

15–6. HARMONIC PROGRESSIONS

A *harmonic progression* is a sequence of non-zero numbers whose reciprocals form an arithmetic progression. Thus, a, b, c are in harmonic progression if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ form an arithmetic progression.

The terms of a harmonic progression between any two given terms are *harmonic means* between the given terms.

To insert a desired number of *harmonic means* between two numbers, we insert the same number of arithmetic means between the reciprocals of the two given numbers and then invert the resulting terms.

The *harmonic mean* of two numbers is found in the following manner. If a, x, b form a harmonic progression, then $\frac{1}{a}, \frac{1}{x}, \frac{1}{b}$ form an arithmetic progression, and $\frac{1}{x}$ is the arithmetic mean of $\frac{1}{a}$ and $\frac{1}{b}$. Hence,

$$\frac{1}{x} = \frac{\frac{1}{a} + \frac{1}{b}}{2}.$$

Solution of this equation for x yields the harmonic mean

$$x = \frac{2ab}{a + b}.$$

Clearly the harmonic mean exists only if $a + b \neq 0$.

Example 15–6. Insert three harmonic means between -3 and 2.

Solution: The corresponding arithmetic progression is $-\frac{1}{3}, \cdots, \frac{1}{2}$. Here $a = -\frac{1}{3}$, $l_5 = \frac{1}{2}$, and $n = 5$. Hence, $\frac{1}{2} = -\frac{1}{3} + 4d$, and $d = \frac{5}{24}$.

It follows that the arithmetic progression is $-\frac{1}{3}, -\frac{1}{8}, \frac{1}{12}, \frac{7}{24}, \frac{1}{2}$.

Therefore, the three harmonic means are $-8, 12, \frac{24}{7}$.

EXERCISE 15–3

In each of the problems from 1 to 6, determine if the given sequence is a harmonic progression. Find the next two terms of the extension of each harmonic progression.

1. $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}$.
2. $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.
3. $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}$.
4. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$.
5. $16, 8, \frac{16}{3}$.
6. $-2, 2, \frac{2}{3}$.
7. Find the tenth term of the harmonic progression $\frac{1}{2}, \frac{1}{7}, \frac{1}{12}, \cdots$.
8. Find the seventh term of the harmonic progression $6, 3, 2, \cdots$.
9. Insert four harmonic means between 1 and 2.

10. Insert three harmonic means between $\frac{1}{3}$ and $\frac{1}{4}\cdot$
11. Insert four harmonic means between 6 and 24.
12. Find the harmonic mean of 6 and 9.
13. Find the harmonic mean of 24 and 72.
14. Insert nine harmonic means between $\frac{2}{3}$ and $\frac{3}{2}\cdot$
15. If a^2, b^2, c^2 form an arithmetic progression, show that $b + c$, $a + c$, $a + b$ form a harmonic progression.
16. If a, b, c form an arithmetic progression and b, c, d form a harmonic progression, show that $ad = bc$.
17. If x is the harmonic mean of a and b, show that $\frac{1}{x - a} + \frac{1}{x - b} = \frac{1}{a} + \frac{1}{b}\cdot$
18. If a, b, c, d form a harmonic progression, show that $\frac{ab}{cd} = \frac{a - b}{c - d}\cdot$
19. If a, b, c form a harmonic progression, show that $\frac{a}{c} = \frac{a - b}{b - c}\cdot$
20. If the harmonic mean of a and b is equal to their arithmetic mean, show that $a = b$, and conversely.

15–7. GEOMETRIC PROGRESSION

A *geometric progression* is a sequence of numbers in which each term after the first is obtained from the preceding one by multiplying it by a fixed number; the multiplier is called the *common ratio.*

The common ratio may be found by dividing any term by the one immediately preceding it. Thus, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ is a geometric progression in which the common ratio is $\frac{1}{8} \div \frac{1}{4} = \frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div 1 = \frac{1}{2}\cdot$ Also, $\sqrt{2}, 1, \frac{1}{\sqrt{2}}, \frac{1}{2}$ is a geometric progression in which the common ratio is $\frac{1}{2} \div \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \div 1 = 1 \div \sqrt{2} = \frac{1}{\sqrt{2}}\cdot$

It follows that a necessary and sufficient condition that three nonzero numbers A, B, and C form a geometric progression is $\frac{C}{B} = \frac{B}{A}\cdot$

15–8. THE GENERAL TERM OF A GEOMETRIC PROGRESSION

Let a denote the first term of a geometric progression, and let r denote the common ratio. Then the progression may be written as follows:

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}.$$

Hence, if l_n represents the value of the nth term,

$$l_n = ar^{n-1} \tag{15-6}$$

15–9. SUM OF THE FIRST n TERMS OF A GEOMETRIC PROGRESSION

Let S_n represent the sum of the first n terms of a geometric progression. Then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}.$$

Multiplying by r, we have

$$S_n r = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n.$$

Subtracting the first of these equations from the second, term by term, we have

$$S_n r - S_n = ar^n - a.$$

Therefore,

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r} \qquad (r \neq 1). \tag{15–7}$$

If we multiply both sides of (15–6) by r, we get $rl_n = ar^n$. Substituting in (15–7), we obtain another useful form for S_n. This is

$$S_n = \frac{a - l_n r}{1 - r} \qquad (r \neq 1). \tag{15–8}$$

Note. If $r = 1$, these formulas do not apply; in this case, however, the geometric progression becomes $a + a + \cdots + a$ to n terms, and $S_n = na$.

Example 15–7. Determine which of the following sequences are geometric progressions:

a) 2, 6, 18; $\qquad b$) 5, 10, 30; $\qquad c$) $x, \dfrac{x^2}{y}, \dfrac{x^3}{y^2}$.

Solution: a) The ratios found by dividing each of the second and third terms by the preceding one are 3 and 3. Hence, the sequence 2, 6, 18 is a geometric progression.

b) In this sequence, the ratios of consecutive terms are 2 and 3. Therefore, the sequence 5, 10, 30 is not a geometric progression.

c) The given sequence is a geometric progression in which the common ratio is x/y.

Example 15–8. Find l_5 and S_5 in the geometric progression 6, 2/3, 2/27, $\cdots$.

Solution: The common ratio is $(2/3) \div 6 = 1/9$. Since $a = 6$ and $n = 5$, we have

$$l_5 = ar^{n-1} = 6(1/9)^4 = \frac{6}{6561}.$$

Also,

$$S_5 = \frac{a(1 - r^n)}{1 - r} = \frac{6(1 - (1/9)^5)}{1 - 1/9} = \frac{6\left(1 - \dfrac{1}{59{,}049}\right)}{1 - 1/9} = \frac{14{,}762}{2187}.$$

Example 15–9. The fifth term of a geometric progression is 3, and the tenth term is -96. Find the common ratio and the first term.

Solution: By the formula (15–6) for the nth term of a geometric progression, we have

$$ar^4 = 3 \quad \text{and} \quad ar^9 = -96.$$

Dividing each side of the second equation by the corresponding member of the first equation, we obtain $r^5 = -32$. Hence, $r = -2$. Therefore, $a(-2)^4 = 3$, or $a = \dfrac{3}{16}$.

Example 15–10. Find each value of x for which the three numbers x, $x - 2$, $x + 1$ form a geometric progression.

Solution: If we apply the condition $\dfrac{C}{B} = \dfrac{B}{A}$, we have $\dfrac{x+1}{x-2} = \dfrac{x-2}{x}$. Solving, we obtain $x = 4/5$. Hence, the geometric progression is $4/5, -6/5, 9/5$.

15–10. GEOMETRIC MEANS

Terms of a geometric progression between any two given numbers are called *geometric means* between the given numbers. Let a and l_n be given numbers. Then k means may be inserted between them by using the formula $l_n = ar^{n-1}$ with $n = k + 2$.

Example 15–11. Insert three geometric means between 1 and 2.

Solution: Let $a = 1$, $l_n = 2$, and $n = 5$. Then $l_5 = ar^4$ and $r = 2^{1/4}$. Hence, the three means are $2^{1/4}, 2^{1/2}, 2^{3/4}$.

If a, x, b form a geometric progression, then x is called *a geometric mean* of a and b. Since $\dfrac{b}{x} = \dfrac{x}{a}$,

$$x = \pm\sqrt{ab}.$$

Hence, we note that a geometric mean of two numbers is the same as a mean proportional between the two numbers.

EXERCISE 15–4

In each of the problems from 1 to 9, determine whether the given sequence is a geometric progression. Find the next two terms of the extension of each geometric progression.

1. $2, 8, 32$. **2.** $\dfrac{1}{2}, -1, 2$. **3.** $4, 16, 64$.

4. $2, 4, 6$. **5.** $27, 18, 12$. **6.** $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{12}$.

7. $\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{6}}{6}, \dfrac{\sqrt{3}}{6}$. **8.** $a, \dfrac{a^2}{2}, \dfrac{a^3}{3}$. **9.** $1, -1, 1$.

In each of the problems from 10 to 15, find l_n and S_n in the given geometric progression.

10. $4, 2, 1, \cdots; n = 10$. **11.** $3, 2, 4/3, \cdots; n = 15$.
12. $3, 9, 27, \cdots; n = 45$. **13.** $100, -10, 1, \cdots; n = 101$.
14. $\log 2, \log 4, \log 16, \cdots; n = 10$. **15.** $\log 9, \log 3, \log \sqrt{3}, \cdots; n = 6$.

16. Find the sum of the first n terms of the geometric progression $1, \dfrac{1}{2}, \dfrac{1}{4}, \cdots$.

17. For what values of x do $x - 2$, $x - 6$, $2x + 3$ form a geometric progression?
18. For what values of x do $3x + 4$, $x - 2$, $5x + 1$ form a geometric progression?
19. For what values of x is $x + 1$ a geometric mean of $2x + 1$ and $x - 1$?
20. Find a geometric mean of 4 and 16. Also, find their arithmetic mean.
21. Find a geometric mean and the arithmetic mean of 3 and 12.
22. Insert four geometric means between 1 and 32.
23. Insert five geometric means between 1 and 1,000,000.
24. Insert ten geometric means between 1 and 2.
25. If A, G, and H denote the arithmetic mean, a geometric mean, and the harmonic mean, respectively, of two numbers a and b, prove that $G^2 = AH$.
26. If the arithmetic mean of a and b, when $a, b \neq 0$, is A, a geometric mean is G, and the harmonic mean is H, find $A - G$, $G - H$, and $A - H$.
27. Show that the sum of the first n terms of the geometric progression $1, \frac{2}{3}, \frac{4}{9}, \cdots$ is $3\left(1 - \left(\frac{2}{3}\right)^n\right)$. Discuss how this sum varies as n increases.
28. Show that the sum of the first n terms of the geometric progression $8, 4, 2, \cdots$ is $16\left(1 - \left(\frac{1}{2}\right)^n\right)$. Discuss how this sum varies as n increases.
29. Find the sum of the first n terms of the sequence $1, 2x, 3x^2, 4x^3, \cdots$. (Hint: Let S_n be the sum. Then compute $S_n - xS_n$.)
30. Find the sum of the terms of the finite sequence $2, \frac{5}{7}, \frac{8}{7^2}, \frac{11}{7^3}, \cdots, \frac{3n+2}{7^n}$.

15–11. INFINITE GEOMETRIC PROGRESSION

A geometric progression in which the number of terms is infinite is called an *infinite geometric progression.*

In Section 15–9, we found an expression for the nth partial sum S_n of a geometric progression. Hence, S_n is the nth term of the *geometric series* based on the given progression. Thus, we have $S_1 = a; S_2 = a + ar; S_3 = a + ar + ar^2; \cdots ; S_n = a + ar + \cdots + ar^{n-1}$. Furthermore,

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad (r \neq 1),$$

or

$$S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \qquad (r \neq 1).$$

Let us consider what happens to the sum of n terms of a geometric progression when the number of terms increases indefinitely.

If $a = 0$, evidently $S_n = 0$ whether $r \neq 1$ or $r = 1$. In this case, the number 0 meets the requirement of a limit of S_n, so that the sum S of the infinite geometric series is equal to 0.

Now suppose $a \neq 0$. For this condition, we consider four cases.

Case 1. Assume that $|r| < 1$. If $r \neq 0$, the numerical value of r^n decreases as n increases. Moreover, by making the number of terms

sufficiently large, we can make $|r^n|$ as small as we please. It follows that if $|r| < 1$, we can make S_n differ from $\frac{a}{1-r}$ by as little as we please; that is, S_n approaches $\frac{a}{1-r}$ as a limit. This condition may be stated symbolically in the following manner:

$$S = \lim_{n \to \infty} S_n = \frac{a}{1-r}, \tag{15-9}$$

where S is the sum of the infinite geometric series. Equation **(15-9)** is true also if $r = 0$, since in this case S_n has the constant value a.

Case 2. If $r = 1$, then $S_n = na$. Since $a \neq 0$, $|S_n|$ increases indefinitely as n increases. Here $\{S_n\}$ diverges.

Case 3. If $|r| > 1$, then $|r^n|$ increases indefinitely as n increases. Hence, so does $|S_n| = \left|\frac{a}{1-r} - \frac{ar^n}{1-r}\right|$. Again $\{S_n\}$ diverges.

Case 4. If $r = -1$, then the progression becomes, $a, -a, a, -a, \cdots, (-1)^{n-1}a$. If n is even, $S_n = 0$. If n is odd, $S_n = a$. In this case, we say that S_n *oscillates* between 0 and a. Here also the series diverges.

The sum of an infinite geometric progression can therefore be found by (15-9), but only when $|r| < 1$. When $r = 1$, $r = -1$, or $|r| > 1$, the series diverges, and we say that the series *has no sum*. For a further discussion of this topic, the student may refer to a treatise on the theory of limits.

Example 15-12. The owner of a fleet of trucks finds that if used motor oil is refined for re-use, 20 per cent of the oil is lost in the process. If he starts with 100 gallons of refined oil and re-refines this oil each time it becomes dirty, determine the total amount of oil he has used before the entire 100 gallons is lost.

Solution: We begin with 100 gallons. After the first reclaiming operation, we have 80 gallons of good oil. When this becomes used and is re-refined, we have 64 gallons; and so on. Theoretically, we would never use up the entire amount of oil. However, the limit of the sum of the amounts of oil reclaimed is approximately reached after a large number of operations. Hence, we have an infinite geometric progression in which $a = 100$ and $r = 4/5$. Then $S = \frac{100}{1 - 4/5} = 500$. Thus, by re-refining the oil as it is used, the fleet owner has had the equivalent of 500 gallons of oil.

15-12. REPEATING DECIMALS

If a decimal contains a fixed sequence of digits which are repeated indefinitely, we call it a *repeating decimal*. Thus, $0.135135\cdots$ is a repeating decimal. This decimal is written $0.\dot{1}3\dot{5}$, the dots indicating the first and last digits of the sequence which is to be repeated. Also, $0.34\dot{5}1\dot{6} = 0.34516516516\cdots$. A repeating decimal is wholly or partly an infinite geometric series. For

example, since $0.34\dot{5}1\dot{6} = 0.34 + .00516 + 0.00000516 + \cdots$, it is composed of the decimal 0.34 and an infinite geometric series in which $a = 0.00516$ and $r = 0.001$. Hence, since $|r| < 1$,

$$0.34\dot{5}1\dot{6} = 0.34 + \frac{0.00516}{1-0.001} = 0.34 + \frac{0.00516}{0.999} = \frac{34}{100} + \frac{172}{33300} = \frac{11494}{33300}.$$

Note that if we divide 172 by 33300, we obtain the repeating decimal $0.00\dot{5}1\dot{6}$.

Example 15–13. Express the repeating decimal $0.2\dot{6}\dot{3}$ as an equivalent numerical fraction.

Solution: We can write this decimal in the form

$$0.2 + 0.063 + 0.00063 + \cdots.$$

Hence, the required number consists of the decimal 0.2 plus an infinite geometric series in which $a = 0.063$ and $r = 0.01$. The sum of the series, since $|r| < 1$, is

$$S = \frac{0.063}{1 - 0.01} = \frac{0.063}{0.99} = \frac{7}{110}.$$

Therefore, $0.2\dot{6}\dot{3} = \frac{1}{5} + \frac{7}{110} = \frac{29}{110}$.

EXERCISE 15–5

In each of the problems from 1 to 12, find the sum of the convergent series based on the given infinite geometric progression.

1. $\frac{8}{5}, -1, \frac{5}{8}, \cdots$. **2.** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$. **3.** $3, \sqrt{3}, 1, \cdots$.

4. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$. **5.** $8, 4, 2, \cdots$. **6.** $8, -4, 2, \cdots$.

7. $1, \frac{2}{3}, \frac{4}{9}, \cdots$. **8.** $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \cdots$. **9.** $1, -\frac{3}{5}, \frac{9}{25}, \cdots$.

10. $0.5, 0.05, 0.005, \cdots$. **11.** $0.18, 0.0018, 0.000018, \cdots$.

12. $0.3 + 0.012 + 0.00012 + 0.0000012 + \cdots$.

13. Find the sum of the series based on the infinite geometric progression $24, 8, \frac{8}{3}, \cdots$. Also, find the sum of the first 20 terms of this progression and compute the error introduced by using S instead of S_{20}.

14. What would be the error if S were used instead of S_n for the sum of the geometric progression $48, -36, 27, \cdots$ to 10 terms?

15. A ball is dropped from a height of 3 feet. On each rebound it bounces back to three-fourths the height from which it last fell. Assuming that this bouncing continues indefinitely, find the distance it travels in coming to rest. How far has it traveled after bouncing ten times?

16. A swinging pendulum will gradually come to rest as a result of friction. If, on each upswing, the pendulum swings through 98 per cent of the arc through which it fell, and if the initial arc for one complete swing was 20 inches, find the distance traveled before the pendulum comes to rest.

Convert each of the following repeating decimals to fractional form.

17. $0.\dot{1}$. **18.** $0.\dot{1}\dot{5}$. **19.** $0.\dot{9}\dot{0}$. **20.** $0.\dot{2}4\dot{3}$.

21. $0.1\dot{6}$. **22.** $0.\dot{1}4285\dot{7}$. **23.** $2.\dot{9}$. **24.** $1.\dot{1}23\dot{4}$.

25. $0.115\dot{4}\dot{2}$. **26.** $2.\dot{1}2\dot{3}$.

15–13. THE BINOMIAL SERIES

We shall now consider the binomial expansion when n is any real number. If we let $a = 1$ and $b = x$ in (4–13), the binomial formula becomes

$$(15\text{–}10)\quad (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r + \cdots.$$

The right member of (15–10) is called a *binomial series.*

We saw in Section 4–6 that, if n is a positive integer, the series on the right in (15–10) terminates with x^n, and (15–10) is true. Otherwise, the terms of the series continue indefinitely, giving rise to an infinite series.

The question which now arises is whether (15–10) is valid when n is not a positive integer; that is, whether the series on the right converges, and, if so, whether its sum is equal to $(1 + x)^n$. It is proved in the study of series that (15–10) is indeed valid if $|x| < 1$. It follows that, if $|x| < 1$, we may obtain the value of $(1 + x)^n$ as accurately as we please by taking sufficiently many terms of the series in (15–10).

The expansion can readily be extended to $(a + b)^n$ when n is not a positive integer. In this case, the expression may be written as follows:

$$(15\text{–}11)\qquad (a + b)^n = \left[a\left(1 + \frac{b}{a}\right)\right]^n = a^n\left[1 + \frac{b}{a}\right]^n.$$

Here $x = \dfrac{b}{a}$, and the expansion of $(a + b)^n$ is valid if $\left|\dfrac{b}{a}\right| < 1$.

Example 15–14. Find the first five terms of the expansion of $(1 + x)^{-3}$ if $|x| < 1$.

Solution: By (15–10),

$$\begin{aligned}(1 + x)^{-3} &= 1 + (-3)x + \frac{(-3)(-4)}{2!}x^2 \\ &\quad + \frac{(-3)(-4)(-5)}{3!}x^3 + \frac{(-3)(-4)(-5)(-6)}{4!}x^4 + \cdots \\ &= 1 - 3x + 6x^2 - 10x^3 + 15x^4 + \cdots.\end{aligned}$$

Example 15–15. Find $\sqrt[3]{1.04}$.

Solution: $\sqrt[3]{1.04} = (1 + 0.04)^{1/3}$. Hence, $n = 1/3$ and $x = 0.04$, and the expansion is

$$(1 + 0.04)^{1/3} = 1 + \frac{1}{3}(0.04) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(0.04)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(0.04)^3 + \cdots.$$

The approximate value of the sum of this series is

$$1 + 0.013333 - 0.000177 + 0.000004 = 1.013160.$$

We have here a case of an *alternating series*, that is, a series in which the terms are alternately positive and negative. It is proved in the study of series that, if in an alternating series each term is numerically less than the preceding term and $\lim_{n\to\infty} a_n = 0$, the error introduced by using S_n as the sum of the series is numerically less than the value of the first term omitted; that is, $|S_n - S| < |a_{n+1}|$. If in the present example we take for S the value $S_3 = 1 + 0.013333 - 0.000177 = 1.013156$, the error in so doing is less than the value of the fourth term 0.000004.

Example 15–16. Find the first four terms of the expansion of $\dfrac{1}{\sqrt[3]{8 - x^2}}$.

Solution: First we apply (15–11) to convert the given expression to a suitable form, as follows:

$$\frac{1}{\sqrt[3]{8 - x^2}} = (8 - x^2)^{-1/3} = \left(8\left(1 - \frac{x^2}{8}\right)\right)^{-1/3}$$
$$= 8^{-1/3}\left(1 - \frac{x^2}{8}\right)^{-1/3} = \frac{1}{2}\left(1 - \frac{x^2}{8}\right)^{-1/3}.$$

Hence, by (15–10), if $x^2 < 8$,

$$\frac{1}{\sqrt[3]{8 - x^2}} = \frac{1}{2}\left(1 + \frac{-1/3}{1}\left(-\frac{x^2}{8}\right) + \frac{(-1/3)(-4/3)}{1\cdot 2}\left(-\frac{x^2}{8}\right)^2 + \frac{(-1/3)(-4/3)(-7/3)}{1\cdot 2\cdot 3}\left(-\frac{x^2}{8}\right)^3 + \cdots\right)$$
$$= \frac{1}{2}\left(1 + \frac{x^2}{24} + \frac{x^4}{288} + \frac{7x^6}{20{,}736} + \cdots\right).$$

EXERCISE 15–6

In each of the problems from 1 to 10, find the first four terms of a binomial expansion of the given expression.

1. $\dfrac{1}{1 + x}$. **2.** $\dfrac{1}{1 - x}$. **3.** $\sqrt{1 + x}$. **4.** $\sqrt{1 - x}$. **5.** $(1 + x)^{-1/2}$.

6. $(1 - x)^{-1/2}$. **7.** $\dfrac{1}{x + y}$. **8.** $\dfrac{1}{(x + y)^2}$. **9.** $\dfrac{1}{(x + y)^3}$. **10.** $(1 + x)^{1/x}$.

Find the approximate value of each of the following numbers by means of a binomial expansion, using four terms of the expression.

11. $(1.02)^{10}$. **12.** $(1.01)^{13}$. **13.** $(1.04)^8$. **14.** $(1.1)^{10}$. **15.** $(0.98)^8$.

16. 49^4. **17.** $(0.99)^6$. **18.** 51^3. **19.** $(1.03)^{1/2}$. **20.** $(0.97)^{-2}$.

16 Mathematical Induction

16–1. METHOD OF MATHEMATICAL INDUCTION

When a certain type of formula or proposition has been verified in specific cases but is not known to be true in general, the method of *mathematical induction* is often found extremely valuable in determining its validity.

Suppose that a statement involving a positive integer n is to be proved true for all values of n greater than or equal to a particular initial value. We begin by showing the result to be true for the first value of n. We then assume that k is some particular integral value of n for which the statement holds. With this assumption as a basis, we establish the validity of the statement in the next succeeding case, namely, that in which $n = k + 1$. In other words, we prove that if the statement is true for any specific integral value of n, say $n = k$, then it is also true for the next larger value of n, namely, $n = k + 1$. Suppose for example, that 1 is the initial value of n. Then the second step establishes that if the statement is true for $n = 1$, it is also true for $n = 1 + 1$, or 2; if it is true for $n = 2$, it is also true for $n = 2 + 1$, or 3; and so on. As a consequence of this, we conclude that the statement is true for all values of n greater than or equal to the initial value, here 1. A proof by mathematical induction, therefore, consists of two parts and a conclusion.

Part 1. Verification that the statement is true for some initial value of n, generally $n = 1$. (This initial value is the smallest value of n for which the statement is to be proved true.)

Part 2. Proof that whenever the statement is true for some particular value of n, say for $n = k$, then it is true for the next larger value of n, that is, for $n = k + 1$.

Conclusion. If both parts of the proof have been given, then the statement is true for all positive integral values of n greater than or equal to the one for which the verification was made in Part 1.

The reasoning process involved here, which consists in taking an initial integer and then repeatedly taking successors, can be exemplified in terms of climbing a ladder. Part 1 puts us on the bottom rung of a ladder. Part 2 shows us how to get from any rung we have reached to the next higher rung. The conclusion states that if we know how to get on the bottom rung of the ladder, and if we know how to get from any rung to the next higher one, then we can reach all rungs, and hence can climb the ladder.

The following examples will illustrate the method.

Example 16–1. If n is any positive integer, prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}. \tag{16–1}$$

Solution:

Part 1: The formula is true when $n = 1$, since

$$\frac{1}{1\cdot 2}=\frac{1}{1+1}, \quad \text{or} \quad \frac{1}{2}=\frac{1}{2}.$$

Part 2: Let k represent any particular value of n for which (16–1) is true. Then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}. \tag{16–2}$$

We now wish to prove that (16–1) is true also for the next larger value of n, namely, $n = k + 1$. The sum on the left in (16–1) when $n = k + 1$ can be obtained by adding its last term, which is $\frac{1}{(k+1)(k+2)}$, to both sides of (16–2). Hence, we have

$$\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}\right)+\frac{1}{(k+1)(k+2)}$$
$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}.$$

But, since $\displaystyle\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}$, we obtain

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}. \tag{16–3}$$

The members of (16–3) are the same as those of (16–1) when $n = k + 1$. Hence, we have shown that (16–3) is true if (16–2) is true, in other words, (16–1) is true for $n = k + 1$ if it is true for $n = k$.

Conclusion: We have shown by verification that (16–1) is true when $n = 1$. Therefore, since (16–1) is true for $n = 1$, it follows from Part 2 that (16–1) is true for every positive integer n.

Example 16–2. Prove that $(x - y)$ is a factor of $(x^n - y^n)$ if n is any positive integer.

Solution:

Part 1: If $n = 1$, then $x^n - y^n = x - y$, which is seen to have $(x - y)$ as a factor.

Part 2: Let k be a specific value of n for which $(x^n - y^n)$ has $(x - y)$ as a factor. We shall now prove that if $(x - y)$ is a factor of $(x^k - y^k)$, it is also a factor of $(x^{k+1} - y^{k+1})$. We have

$$x^{k+1} - y^{k+1} = x^{k+1} - xy^k + xy^k - y^{k+1} = x(x^k - y^k) + y^k(x - y).$$

By assumption, $(x - y)$ is a factor of $(x^k - y^k)$. Also, by inspection, $(x - y)$ is a factor of $y^k(x - y)$. Hence, $(x - y)$ is a factor of the left member $(x^{k+1} - y^{k+1})$. Therefore, if the conclusion is true for $n = k$, it is also true for $n = k + 1$.

Conclusion: By Part 1 of the proof, $(x - y)$ is a factor of $(x^n - y^n)$ when $n = 1$. Therefore, by virtue of Part 2, the desired conclusion is true for any positive integer n.

16–2. PROOF OF THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS

We shall now prove that the binomial formula,

$$(16\text{–}4) \quad (a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1} + \frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}b^r + \cdots + b^n,$$

is true for every positive integral value of n.

Proof. *Part 1.* When $n = 1$, each side of (16–4) becomes $a + b$. Hence, (16–4) is true for $n = 1$.

Part 2. Let k be any specific value of n for which (16–4) is true. Then we have

$$(a + b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \cdots + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+1}b^{r-1} + \frac{k(k-1)\cdots(k-r+1)}{r!}a^{k-r}b^r + \cdots + b^k.$$

Multiplying each member of this equation by $(a + b)$, we obtain

$$(a + b)^{k+1} = a^{k+1} + ka^kb + \cdots + \frac{k(k-1)\cdots(k-r+1)}{r!}a^{k-r+1}b^r + \cdots + ab^k + a^kb + \cdots + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+1}b^r + \cdots + b^{k+1}$$

Hence, by combining terms, we get

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \cdots + \frac{(k+1)k\cdots(k-r+2)}{r!}a^{k-r+1}b^r + \cdots + b^{k+1}.$$

In this result the sum of the coefficients of $a^{k-r+1}b^r$ is obtained as follows:

$$\frac{k(k-1)\cdots(k-r+2)(k-r+1)}{r!} + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}$$
$$= \left[\frac{k-r+1}{r} + 1\right]\cdot\left[\frac{k(k-1)\cdots(k-r+2)}{(r-1)!}\right]$$
$$= \frac{(k+1)(k)(k-1)\cdots(k-r+2)}{r!}.$$

We note that the value of $(a+b)^{k+1}$, obtained as the product of $(a+b)^k$ and $(a+b)$, is exactly the same as the expansion which would be obtained from (16–4) with $n = k+1$. Hence, we have shown that if the binomial formula holds for $n = k$, it must hold for $n = k+1$.

Conclusion. The binomial formula was seen to be true for $n = 1$. Therefore, by virtue of Part 2, we may conclude that it is true for every positive integer n.

EXERCISE 16–1

Prove by mathematical induction that each of the statements in Problems 1 to 20 is true for all positive integral values of n.

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

2. $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

3. $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

4. $1 + 3 + 6 + \cdots + \dfrac{n(n+1)}{2} = \dfrac{n(n+1)(n+2)}{6}$.

5. $3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n+1)}{2}$.

6. $4 + 8 + 12 + \cdots + 4n = 2n(n+1)$.

7. $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$.

8. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n} = 1 - \left(\dfrac{1}{2}\right)^n$.

9. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

10. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

11. $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \dfrac{2n(n + 1)(2n + 1)}{3}$.

12. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n + 1)}{2}\right]^2$.

13. $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$.

14. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n + 1)^2$.

15. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

16. $3 + 3^2 + 3^3 + \cdots + 3^n = \dfrac{3}{2}(3^n - 1)$.

17. $4 + 4^2 + 4^3 + \cdots + 4^n = \dfrac{4}{3}(4^n - 1)$.

18. $1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n - 1)(2n + 1)}{3}$.

19. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$.

20. $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{1}{3 \cdot 4 \cdot 5} + \cdots + \dfrac{1}{n(n + 1)(n + 2)} = \dfrac{n(n + 3)}{4(n + 1)(n + 2)}$.

21. Prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for every positive integer n.

22. Prove that $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$, for every positive integer n.

23. By using mathematical induction, prove the formula for the sum of an arithmetic progression.

24. By using mathematical induction, prove the formula for the sum of a geometric progression.

17 Permutations, Combinations, and Probability

17–1. FUNDAMENTAL PRINCIPLE

We begin the study of permutations and combinations by considering the following principle, which is fundamental for the entire subject.

Fundamental Principle. If one thing can be done in a ways, and if, for each such way, a second thing can be done in b ways, then the two together can be done in $a \cdot b$ ways.

To understand why the principle is true, note that for each of the a ways of doing the first thing, there are b ways of doing the second; hence, both the first and the second things taken together can be done in $a \cdot b$ ways.

The following examples will illustrate the reasoning upon which the principle is based, as well as an obvious extension of the principle to the case when more than two things are to be done.

Example 17–1. In how many ways can two officers, a chairman and a secretary, be selected from a committee of five men?

Solution: By the fundamental principle, the problem is equivalent to determining the number of ways in which the two things can be done together. The first position can be filled in 5 ways; that is, there are $a = 5$ ways of selecting a chairman. For each of these possible selections, there are $b = 4$ ways of filling the position of secretary from the remaining men. Hence, the number of ways of selecting a chairman and secretary is $a \cdot b = 5 \cdot 4 = 20$.

Example 17–2. How many three-digit numbers can be formed from the ten digits $0, 1, 2, \cdots, 9$, if *a*) repetitions of digits are not permitted; *b*) repetitions are permitted?

Solution: *a*) Here we have three things to do or places to fill. The hundreds place can be filled in 9 ways, since 0 must be excluded from this place. The tens

place can then be filled in 9 ways from any of the remaining 9 digits. Finally, the units place can be filled from any of the remaining 8 digits. There are, therefore, $9 \cdot 9 \cdot 8 = 648$ three-digit numbers in which no two digits are alike.

b) If repetitions are permitted, there are 9 ways to fill the hundreds place and 10 ways to fill each of the tens and units places. Hence, there are $9 \cdot 10 \cdot 10 = 900$ three-digit numbers.

EXERCISE 17–1

1. Nine persons apply for each of two vacant apartments. In how many possible ways can both apartments be rented?
2. A large room has eight doors. In how many ways can a person enter the room by one door and leave by a different door?
3. If three dice are thrown, in how many ways can they fall?
4. There are eight men and six women in a club. In how many ways can two officers be selected so that one is a man and one is a woman?
5. How many possible four-digit numbers are there in a telephone exchange which uses only four-digit numbers and excludes 0000?
6. How many six-digit numbers can be formed from the digits 2, 3, 4, 5, 6, 7, 8, 9? How many of these are larger than 700,000?
7. How many numbers of six different digits can be formed from the digits 2, 3, 4, 5, 6, 7, 8, 9? How many of these are larger than 700,000?
8. A woman has seven guests at a party. If she chooses her seat first, in how many ways can she seat her guests?

17–2. PERMUTATIONS

An ordered arrangement of all or any part of a set of things is called a *permutation*. Specifically, suppose we have n distinct things and wish to select r of these to be arranged in a definite order. Each such ordered arrangement is called a *permutation of n things r at a time*. The number of all such permutations is denoted by ${}_nP_r$. Thus, ${}_5P_2$ is read "the number of permutations of 5 different things 2 at a time."

In Example 17–1, the possible number of ways of selecting a chairman and secretary from a committee of five men was found by means of the fundamental principle to be $5 \cdot 4 = 20$. This is precisely equal to ${}_5P_2$, since it is the number of ways in which two men can be chosen from among the given five men and arranged in the two offices. The number of permutations of n things r at a time is given by the formula

$$(17\text{–}1) \qquad {}_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

The truth of (17–1) is readily shown as follows. The first of the r places can be filled in n ways. Then the second can be filled in $(n-1)$ ways, the third in $(n-2)$ ways, and so on. In general, the

number of ways of filling each place is n minus the number of places already filled. Therefore, when the rth object is chosen, $(r-1)$ places have already been filled, and the rth place can then be filled in $n-(r-1)$, or $n-r+1$, ways.

In particular, if $r=n$, the last factor becomes $n-n+1=1$. We then have

$$(17\text{–}2)\qquad {}_nP_n = n(n-1)(n-2)\cdots 1 = n!.$$

This formula gives the number of permutations of n different things taken all at a time.

If both the numerator and the (understood unit) denominator of (17–1) are multiplied by $(n-r)!$, we obtain the following alternate formula:

$$(17\text{–}3)\quad {}_nP_r = \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}\,.$$

Here it is agreed that by definition $0\,!=1$. Hence, (17–3) holds for $r=n$.

For example, by (17–2),

$${}_8P_5 = n(n-1)(n-2)\cdots(n-r+1) = 8\cdot7\cdot6\cdot5\cdot4 = 6720.$$

Using (17–3), we have also

$${}_8P_5 = \frac{n!}{(n-r)!} = \frac{8!}{3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3} = 6720.$$

17–3. PERMUTATIONS OF n THINGS NOT ALL DIFFERENT

Suppose it is required to find the number of indistinguishable distinct permutations of n things all at a time, if n_1 things are regarded as indistinguishable, n_2 other things are regarded as indistinguishable, and so on. Let us consider, for example, the number of permutations of the letters a, a, a, b taken four at a time. For convenience, indistinguishable objects are given the same notation.

Denote by P the desired number of distinct permutations. Evidently, P is less than ${}_4P_4$, which is the number of permutations that could be effected if all the letters were distinguishable. For in any one of the P permutations, say $(a\ b\ a\ a)$, any rearrangement of the a's among themselves would not change the permutation. If, however, we assigned subscripts to a in this permutation, as in $(a_1\ b\ a_2\ a_3)$, we could permute these three distinct letters among themselves in $3!$ ways. This can be done for each of the P permutations of the letters a, a, a, b. We would then obtain $P\cdot 3!$ permutations of the four distinct letters a_1, a_2, a_3, b taken four at a time. There are therefore $4!$ permutations altogether. Hence,

$$P\cdot 3! = 4!,\quad\text{or}\quad P = \frac{4!}{3!} = \frac{4!}{3!\,1!}\,.$$

In general, the number of distinct permutations of n things taken all at a time, if n_1 things are alike, n_2 other things are alike, n_3 other things are also alike, and so on, equals

$$(17\text{–}4) \qquad P = \frac{n!}{n_1!\, n_2!\, n_3! \cdots}.$$

17–4. COMBINATIONS

A *combination* is a set of all or any part of a collection of objects without regard to the order of the objects in the set. We use the symbol ${}_nC_r$ to represent the total number of all combinations of n different things taken r at a time.

The different sets of the four letters a, b, c, d taken three at a time, without reference to the order in which the letters are arranged, are (abc), (abd), (acd), (bcd). From each of these four combinations, we can form $3!$, or 6, different permutations of the four letters taken three at a time. For example, from the combination (abc) we can form the distinct permutations (abc), (acb), (bac), (bca), (cab), (cba). Hence, each of the four combinations contributes ${}_3P_3 = 3!$ permutations to the total number of permutations. Thus, there are ${}_4C_3$ combinations of the four letters if we disregard order, and there are $3!$ ordered arrangements or permutations of each combination. Hence, we have ${}_4P_3 = 3! \cdot {}_4C_3$, or ${}_4C_3 = \frac{{}_4P_3}{3!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} = 4$ combinations. This is the same as the number we obtained at the beginning of the paragraph.

In general, if we divide the total number of permutations of n things r at a time, or ${}_nP_r$, by the number of permutations, or $r!$, contributed by each combination, we obtain the total number of combinations. Symbolically, we have

$${}_nP_r = r! \cdot {}_nC_r,$$

or

$$(17\text{–}5) \qquad {}_nC_r = \frac{{}_nP_r}{r!}.$$

If we note that $r! = {}_rP_r$, then we can write the following interesting relationship:

$${}_nC_r = \frac{{}_nP_r}{{}_rP_r}.$$

Replacing ${}_nP_r$ by its equivalent expression from (17–1) or (17–3) we have

$$(17\text{–}6) \qquad {}_nC_r = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}.$$

If r is replaced by $(n - r)$ in (17–6), we obtain

$$_nC_{n-r} = \frac{n!}{(n-r)!\,r!}.$$

Hence,

$$_nC_r = {}_nC_{n-r}. \tag{17-7}$$

Note. Having agreed that $0! = 1$, we can supply (17–1) or (17–3) to permutations or combinations of n things zero at a time. Considering (17–5), we have

$$_nC_0 = \frac{n!}{0!\,(n-0)!} = \frac{1}{0!} = 1. \tag{17-8}$$

This result agrees with the intuitive conclusion that there is only one such "empty" combination. Similarly, $_nP_0 = 1$.

17–5. BINOMIAL COEFFICIENTS

By referring to the development of the binomial formula in Section 4–6, we note that the expansion of $(a + b)^n$ involves the product $(a + b)(a + b)(a + b)\cdots$ taken without regard to order. This fact suggests the use of combinations in the coefficients of the expansion.

We said in Section 4–7 that the coefficient of the term involving $a^{n-r}b^r$ is $\dfrac{n(n-1)\cdots(n-r+1)}{r!}$. This is precisely the formula for the combination of n things r at a time, or $_nC_r$, obtained in Section 17–4. The binomial formula may therefore be written as follows:

$$(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots + {}_nC_nb^n. \tag{17-9}$$

For example, the expansion of $(a + b)^4$ takes the following form:

$$_4C_0a^4 + {}_4C_1a^3b + {}_4C_2a^2b^2 + {}_4C_3ab^3 + {}_4C_4b^4.$$

This reduces to

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

If we let $a = b = 1$ in the expansion for $(a + b)^n$ in (17–9), we obtain

$$(1 + 1)^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n.$$

Since $_nC_0 = 1$, $_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1$.

Thus, the total number of combinations of n things taken successively $1, 2, \cdots, n$ at a time is $2^n - 1$.

EXERCISE 17–2

1. Evaluate $_5P_2$, $_7P_3$, $_{12}P_5$, $_{21}P_4$, $_{100}P_3$.
2. Evaluate $_{10}C_4$, $_{11}C_3$, $_{100}C_5$, $_{21}C_5$, $_{100}C_{95}$.

3. Form all possible distinct permutations of the letters of the word *theory*. **a.** How many are there? **b.** How many begin and end with a vowel? **c.** How many begin or end with a vowel?
4. How many distinct permutations are there of the five letters a, b, c, d, e taken three at a time? Write them out.
5. In how many different ways can a dime, a quarter, and a half-dollar be distributed among five boys?
6. How many different sums can be formed with a penny, a nickel, a dime, and a quarter?
7. Expand $(a + b)^8$, using combination symbols.
8. From the six digits 1, 2, 3, 4, 5, 6, form all permutations taken five at a time. **a.** How many are formed? **b.** How many begin with 4? **c.** In how many does the digit 3 not appear?
9. How many distinct permutations can be made from the letters of the word *probability* taken all at a time?
10. How many different combinations are there of 4 identical nickels, 5 identical dimes, and 6 identical quarters?
11. How many different straight lines are determined by twelve points, no three of which are in a straight line?
12. In how many different ways can a student select seven questions out of ten on a test?
13. How many different weights can be formed with six objects weighing 1, 2, 4, 8, 16, and 32 pounds, respectively?
14. In how many different ways can signals be made with seven different flags, where a signal is a set of one or more flags arranged in a specific order?
15. In how many different ways can the 52 cards of a bridge deck be dealt among four players?

17–6. MATHEMATICAL PROBABILITY

If, in a given trial, an event can happen in h different ways and can fail to happen in f ways, and if all the $h + f$ ways are equally likely, then the probability p that the event will happen in this trial is

$$p = \frac{h}{h+f}. \tag{17–10}$$

The probability q that the event will fail to happen is

$$q = \frac{f}{h+f}. \tag{17–11}$$

Note that $0 \leqq p \leqq 1$, $0 \leqq q \leqq 1$, and $p + q = 1$. Two illustrations of probability follow.

If a bag contains 3 green marbles and 4 yellow marbles, all exactly alike except for color, the probability of drawing a single green marble is

$$p = \frac{3}{3+4} = \frac{3}{7}.$$

A die can fall in six ways. The probability of getting 5 or more with one throw of a die is $\frac{2}{6}$, or $\frac{1}{3}$, since there are two possible ways, either a 5 or a 6, for the event to happen.

17–7. MOST PROBABLE NUMBER AND MATHEMATICAL EXPECTATION

Let p be the probability of occurrence of some event. Furthermore, suppose that n trials of the event are made, of which h are successful. Then $\frac{h}{n}$ is called the *relative frequency* of success for the trials which occurred. It is not to be expected that $\frac{h}{n} = p$. It is shown in more advanced treatments of probability, however, that if n is large, it is very likely that the relative frequency is approximately equal to p. Also, the larger we take n, the more likely it is that $\frac{h}{n}$ approximates p closely. Moreover, it can be shown that the most probable or expected number of occurrences of the event for n trials is np.

For example, when a coin is tossed, the probability of getting a head is 1/2. In 1,000 trials the expected number of heads is therefore 500. This does not mean, however, that if the first 100 trials result in 75 heads and 25 tails, we should expect 25 heads and 75 tails in the next 100 trials. Actually, since one toss of the coin does not affect the next one, we should expect about 50 heads and 50 tails in the next 100 trials. Moreover, we may expect about 450 heads and 450 tails in the next 900 trials.

If p is the probability of winning a certain amount of money in case a certain event occurs and m is the amount of money to be won, the *mathematical expectation* is defined to be pm. For instance, if a person can win \$12 provided he throws an ace with a die, his expectation is $\frac{1}{6}(\$12) = \$2$. Hence, \$2 is the fair amount he should be willing to pay to make the trial.

17–8. STATISTICAL, OR EMPIRICAL, PROBABILITY

It is frequently impossible to have sufficient knowledge beforehand of all the conditions that might cause an event to happen or fail to happen. In such a case, however, it may be possible to determine the relative frequency of the occurrence of the event from a large number of trials. Thus, if an event has been observed to happen h times in n trials, and n is a large number, then until addi-

tional knowledge is available, we define the *statistical probability*, or *empirical probability*, to be

$$p = \frac{h}{n}, \tag{17-11}$$

where $\frac{h}{n}$ is the relative frequency.

Example 17–3. A molding machine turns out 12 parts per minute. Inspection experience has shown that there are 20 defective parts per hour. What is the probability that a single part, picked at random, will be defective? In a run of 10,000 parts, how many defectives should be expected?

Solution: The parts are produced at the rate of 720 per hour, and 20 of them are defective. Hence, the probability that a single part selected at random will be defective is $\frac{20}{720} = \frac{1}{36}$. In a run of 10,000 parts, we should expect $\frac{1}{36}$ (10,000), or approximately 278, defective parts.

17–9. MUTUALLY EXCLUSIVE EVENTS

Two or more events are *mutually exclusive* if not more than one of them can happen in a given trial. The following theorem may be stated.

Theorem. The probability that some one of a set of mutually exclusive events will happen in a given trial is the sum of the individual-event probabilities.

Proof. Consider, for simplicity, a set of two mutually exclusive events. Suppose that the first can happen in h_1 ways and the second can happen in h_2 ways, and let n be the total number of ways in which the two events can happen or fail to happen. Then $p_1 = \frac{h_1}{n}$ and $p_2 = \frac{h_2}{n}$ are the corresponding probabilities of the two events.

Since the n events are mutually exclusive, the h_1 ways are different from the h_2 ways, and the number of ways that either the first or the second event can happen is therefore $h_1 + h_2$. Hence, the probability p that either the one event or the other will happen is

$$p = \frac{h_1 + h_2}{n} = \frac{h_1}{n} + \frac{h_2}{n} = p_1 + p_2. \tag{17-12}$$

For example, suppose that a bag contains 2 green marbles, 3 yellow marbles and 5 brown marbles. If a marble is drawn at random, the probability that it is green is $\frac{2}{10}$, and the probability that it is yellow is $\frac{3}{10}$. Hence if a marble is drawn, the probability that it is either green or yellow is $\frac{2}{10} + \frac{3}{10}$, or $\frac{5}{10}$, or $\frac{1}{2}$.

17–10. DEPENDENT AND INDEPENDENT EVENTS

In case two or more events are not mutually exclusive, they are dependent if the occurrence of any one affects the occurrence of the others, and they are independent if the occurrence of one does not affect the occurrence of the others.

For example, if a card is drawn from a deck of 52 cards and the card is not replaced before a second is drawn, then the second drawing is *dependent* on the outcome of the first. If, however, the first card is replaced, then the second drawing is *independent* of the first. In the latter case the two drawings are equivalent to simultaneous drawings from two decks.

We shall now state and prove the following theorem relating to dependent and independent events.

Theorem. The probability that two dependent or independent events will occur (successively if dependent; successively or simultaneously if independent) is the product of their individual probabilities.

Proof. Suppose that the first event can happen in h_1 out of a total of n_1 different ways, and that the second event can happen in h_2 out of n_2 different ways. Then it follows, by the fundamental principle in Section 17–1, that the two events can happen together in h_1h_2 ways out of a total of n_1n_2 different ways. Therefore, the probability that both events will happen is

$$p = \frac{h_1h_2}{n_1n_2} = \frac{h_1}{n_1}\cdot\frac{h_2}{n_2} = p_1p_2. \tag{17–13}$$

Example 17–4. Two cards are drawn from a deck containing 52 cards. Find the probability that both cards are aces when the first card is not replaced before the second is drawn.

Solution: We shall begin with a listing of the following useful probabilities:

1) The probability of drawing an ace from a deck of 52 cards is $\frac{4}{52}\cdot$
2) If the first card is an ace and it is not replaced, the probability of drawing another ace is $\frac{3}{51}\cdot$
3) If the first card is not an ace and is not replaced, the probability of the second being an ace is $\frac{4}{51}\cdot$
4) If the first card is replaced, the probability of the second being an ace is $\frac{4}{52}\cdot$ This drawing is entirely independent of the first drawing.

Consider, now, the given problem. The probability that the first card is an ace is $p_1 = \frac{4}{52} = \frac{1}{13}\cdot$ If the first card drawn is an ace, then the probability that the

second card drawn is an ace is $p_2 = \frac{3}{51} = \frac{1}{17}\cdot$ Hence, the probability that both will be aces is $p_1p_2 = \frac{1}{13}\cdot\frac{1}{17} = \frac{1}{221}\cdot$

17–11. REPEATED TRIALS

Theorem. If p is the probability that an event will happen in any trial, and $q = 1 - p$ is the probability that it will fail, then the probability that it will happen exactly r times out of n trials is

$$(17\text{–}14) \qquad {}_nC_r p^r q^{n-r} = \frac{n!}{r!\,(n-r)!}\,p^r q^{n-r}.$$

Proof. The happening of the event in exactly r trials and its failure in the remaining $n - r$ times are independent events. Hence, the probability, by the theorem in Section 17–10, is $p^r q^{n-r}$. But these r trials can be chosen from the n trials in ${}_nC_r$ ways. Since these ways are mutually exclusive, the total probability is ${}_nC_r p^r q^{n-r}$. Note that this expression is the $(r + 1)$th term of the binomial formula for $(q + p)^n$, since

$$(q + p)^n = q^n + {}_nC_1 q^{n-1}p + {}_nC_2 q^{n-2}p^2 + \cdots + {}_nC_r q^{n-r}p^r + \cdots + p^n.$$

The successive terms of this expansion give the probabilities that the event will happen exactly $0, 1, 2, \cdots, r, \cdots n$ times in n trials.

An event will happen *at least* r times in a given number of n trials if it happens $n, n - 1, \cdots, r + 1$, or r times. Since these events are mutually exclusive, the probability that an event will happen at least r times is given by the sum

$$p^n + {}_nC_{n-1}\, qp^{n-1} + \cdots + {}_nC_r\, q^{n-r}p^r.$$

Example 17–5. What is the probability of tossing an ace exactly three times in four trials with one die?

Solution: Since the probability of tossing an ace in one trial is $\frac{1}{6}$ and the probability of failure is $\frac{5}{6}$, we may substitute in the term ${}_nC_r q^{n-r}p^r$ of the binomial formula. The result is

$${}_4C_3 \left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)^3 = 4\left(\frac{5}{6}\right)\left(\frac{1}{216}\right) = \frac{5}{324}\cdot$$

Example 17–6. What is the probability of tossing an ace at least twice in four trials with one die?

Solution: The event will happen at least twice if it happens 4, 3, or 2 times. Hence, the probability is given by the following sum:

$${}_4C_4 \left(\frac{1}{6}\right)^4 + {}_4C_3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3 + {}_4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = \frac{171}{1296}\cdot$$

EXERCISE 17–3

1. A certain event can happen in four ways and can fail to happen in six ways. What is the probability that it will happen? If $60 can be won on the event, what is the mathematical expectation?
2. A box contains 5 white balls, 4 red balls, and 13 black balls. **a.** If one ball is drawn out, what is the probability that it is red? **b.** What is the probability that it is white or red?
3. **a.** If one die is thrown, what is the probability that a "1" or a "2" will turn up? **b.** What is the probability that a "3" or larger number will turn up?
4. When a coin was tossed 100 times, 80 heads and 20 tails turned up. If the tossing were continued until 200 tosses had been made, what would be the most probable number of tails in the second 100 tosses?
5. A bag contains five $1 bills, ten $5 bills, and twenty $10 bills. If one bill is drawn, what is the mathematical expectation?
6. What is the probability of throwing a "7" or an "11" on one throw of two dice?
7. An automobile owner carries $1,000 theft insurance on his car. If, during the past year, 237 out of 97,864 automobiles registered in his area were stolen, what is the mathematical value of the policy?
8. In a city of 77,000 families, a careful sample of 800 families showed that 120 of the sample families owned their own homes. **a.** What is the probability that a family selected at random in the city owned its home? **b.** What is the expected number of families in the city who own their own homes?
9. In a certain city 28,600 persons voted for one candidate for an office, and 23,100 voted for his opponent. What is the probability that a voter chosen at random voted for the winner?
10. A bag contains 5 red balls and 9 black balls. If two balls are drawn in succession, and the first is not replaced, find the probability that the first is red and the second is black.
11. In a baseball tournament the probability that team A will win is $\frac{1}{7}$, and the probability that team B will win is $\frac{1}{9}$. Find the probability that one of these two teams will win.
12. The probability that team A will reach the finals of a tournament is $\frac{2}{7}$, and the probability that it will win the finals is $\frac{1}{3}$. Find the probability that team A will win the tournament.
13. Find the probability of throwing three successive fours on a pair of dice.
14. The probability of A winning a game when he plays it is $\frac{1}{4}$. He is scheduled to play four times. **a.** Find the probability that he will win exactly three times. **b.** Find the probability that he will win at least three times.
15. Three dice are tossed. **a.** Find the probability that exactly two threes will turn up. **b.** What is the probability that at most two threes will turn up?

18 Solution of the General Triangle

18–1. CLASSES OF PROBLEMS

There are certain relationships among the lengths of the sides and the trigonometric functions of the angles of every triangle. If one side and any two other parts of a triangle are given, the remaining parts can be determined; that is, the triangle can be solved. The three given parts may comprise any one of the following four combinations:

Case I. One side and two angles.

Case II. Two sides and the angle opposite one of them.

Case III. Two sides and the included angle.

Case IV. Three sides.

In this chapter, we shall discuss methods for treating these four cases. For convenience, we shall let ABC denote any triangle whose angles are A, B, and C; and we shall let a, b, and c represent the lengths of the corresponding opposite sides.

18–2. THE LAW OF SINES

Law of Sines. Let ABC be any triangle lettered in the conventional manner. Then the following relationship between the sides and the sines of the angles may be written:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{18–1}$$

This relationship is commonly called the *law of sines.*

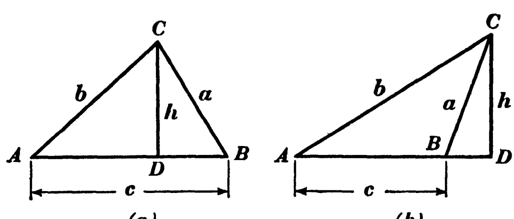

FIG. 18–1.

Proof. We first note that all angles may be acute, as in Fig. 18–1(a), or one angle, say B, may be obtuse, as in Fig. 18–1(b). (The case where $B = 90°$ entails no difficulties and will therefore be omitted.) In each diagram, let h denote the altitude from the vertex C to the side AB. Then, in either case, $\sin A = \frac{h}{b}$ and $\sin B = \frac{h}{a}\cdot$ Dividing the first equation by the second, we have

$$\frac{\sin A}{\sin B} = \frac{a}{b}, \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B}\cdot$$

In a similar way, by drawing the altitude from the vertex A to the side BC, we get

$$\frac{b}{\sin B} = \frac{c}{\sin C}\cdot$$

The equations thus obtained may be combined to give the law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\cdot \tag{18–1}$$

Note. The law of sines is well adapted to the use of logarithms because it involves only multiplications and divisions.

Since any pair of ratios in the law of sines involves two angles and the sides opposite, it may be used in the solution of problems in Cases I and II.

As noted above, the law of sines also applies in the special case where ABC is a right triangle. In this case, one of the angles is $90°$, and the sine of that angle is 1.

18–3. SOLUTION OF CASE I BY THE LAW OF SINES: GIVEN ONE SIDE AND TWO ANGLES

When one side and any two angles of a triangle are known, the third angle can be found from the relation $A + B + C = 180°$, and each of the required sides is uniquely determined. These sides may be found by the law of sines.

Example 18–1. In a triangle ABC, $A = 38°14'$, $B = 67°20'$, $c = 329$. Solve the triangle.

Solution: The values of the given parts are indicated in Fig. 18–2. In this case,

$$C = 180° - (38°14' + 67°20') = 74°26'.$$

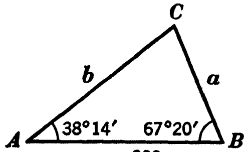

FIG. 18–2.

To find a, we use the relationship

$$\frac{a}{\sin 38°14'} = \frac{329}{\sin 74°26'}\cdot$$

Therefore,

$$a = \frac{329 \sin 38°14'}{\sin 74°26'},$$

and

$$\log a = \log 329 + \log \sin 38°14' - \log \sin 74°26'.$$

The indicated operations may be performed as follows:

$$\begin{aligned} \log 329 &= 2.5172 \\ \log \sin 38°14' &= \underline{9.7916 - 10} \\ &\ 12.3088 - 10 \\ \log \sin 74°26' &= \underline{9.9838 - 10} \\ \log a &= 2.3250 \\ a &= 211. \end{aligned}$$

To determine b, we use the relationship

$$\frac{b}{\sin 67°20'} = \frac{329}{\sin 74°26'}.$$

Hence,

$$b = \frac{329 \sin 67°20'}{\sin 74°26'},$$

and

$$\log b = \log 329 + \log \sin 67°20' - \log \sin 74°26'.$$

The work follows:

$$\begin{aligned} \log 329 &= 2.5172 \\ \log \sin 67°20' &= \underline{9.9651 - 10} \\ &\ 12.4823 - 10 \\ \log \sin 74°26' &= \underline{9.9838 - 10} \\ \log b &= 2.4985 \\ b &= 315. \end{aligned}$$

As shown in Fig. 18–1(a),

$$c = b \cos A + a \cos B.$$

This relationship may be used as a check. Thus,

$$\begin{aligned} c &= 315 \cos 38°14' + 212 \cos 67°20' \\ &= (315)\ (0.7855) + (212)\ (0.3854) = 329.1, \end{aligned}$$

which agrees satisfactorily with the given value of c.

18–4. SOLUTION OF CASE II BY THE LAW OF SINES GIVEN TWO SIDES AND THE ANGLE OPPOSITE ONE OF THEM

Case II is called the *ambiguous case,* because the data may be such that two, one, or no triangles are determined. The number of solutions when a, b, and A are given is indicated by the accompanying table.

TABLE OF POSSIBLE SOLUTIONS

(18–2)	A acute	$a = b \sin A$	One right triangle
(18–3)		$a < b \sin A$	No triangle
(18–4)		$b \sin A < a < b$	Two triangles
(18–5)		$a \geqq b$	One triangle
(18–6)	A obtuse	$a \leqq b$	No triangle
(18–7)		$a > b$	One triangle

We shall use Fig. 18–3 to illustrate in turn the different possibilities considered in the table. If the angle A and the sides a and b are given, we first construct the angle A with the initial ray AX and the terminal ray AR. Next, we lay off the distance $AC = b$ along the terminal side. Then, with C as the center and the length of the side a as the radius, we describe an arc. We mark the point or points in which this arc intersects the initial ray AX of the angle A.

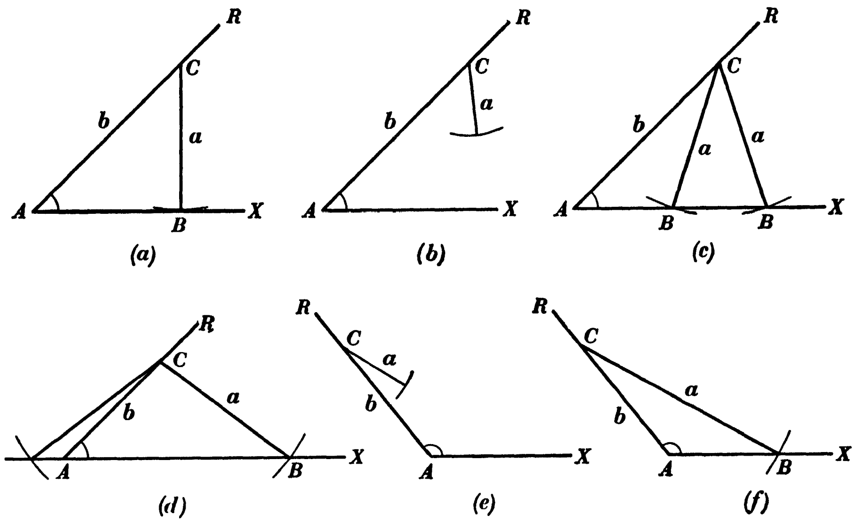

FIG. 18–3.

Figure 18–3(a) corresponds to (18–2) in the table. Since $BC = a = b \sin A$, this segment is the altitude of the triangle drawn from the vertex C. Hence, the arc with radius a is tangent to the initial side at B, and the triangle is a right triangle.

Figure 18–3(b) corresponds to (18–3) in the table. Since $a < b \sin A$, the side a is too short to intersect AX, and there is no triangle.

In Fig. 18–3(c), $a < b$ and $b \sin A < a$, as stated in (18–4) in the table, and the arc will intersect AX in two points marked B and B'. Therefore, two solutions exist. The angle B' in the triangle $AB'C$ is the supplement of the angle B in the triangle ABC.

Figure 18–3(d) represents the case in which the side a is longer than the side b, as stated in (18–5) in the table. Hence, there is only one point B in which the arc with radius a intersects the initial ray AX of the angle A. There is only one solution.

In Fig. 18–3(e), the angle A is obtuse. Since $a < b$, as stated in (18–6), the radius a is too short to intersect the initial ray AX, and no triangle exists.

Finally, in Fig. 18–3(f), A is obtuse and $a > b$, as stated in (18–7). Here the arc can intersect the initial ray AX in only one point. There is, in this case, only one triangle.

The following examples will illustrate some of the possibilities.

Example 18–2. In a triangle ABC, $A = 36°15'$, $a = 9.8$, $b = 12.4$. Solve the triangle.

Solution: Draw Fig. 18–4 approximately to scale, showing the given parts. After angle A and side b have been drawn, an arc is described with C as center and a as radius. The arc intersects the side AX in two points, B_1 and B_2, and we apparently have two possible triangles, AB_1C and AB_2C.

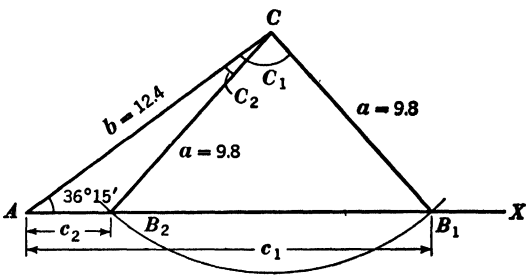

FIG. 18–4.

To find $\sin B$, we use

$$\frac{\sin B}{b} = \frac{\sin A}{a}.$$

Then

$$\sin B = \frac{12.4 \sin 36°15'}{9.8},$$

and

$$\log \sin B = \log 12.4 + \log \sin 36°15' - \log 9.8$$

The logarithmic work follows:

$$\begin{aligned}
\log 12.4 &= 1.0934 \\
\log \sin 36°15' &= \underline{9.7718 - 10} \\
& 10.8652 - 10 \\
\log 9.8 &= \underline{0.9912} \\
\log \sin B &= 9.8740 - 10 \\
B &= 48°26'.
\end{aligned}$$

There are two solutions, since $b \sin A < a < b$. The left inequality follows from the fact that $\log \sin B = \log \dfrac{b \sin A}{a} < 0$, and so $\dfrac{b \sin A}{a} < 1$. If we let $B_1 = 48°26'$,

then $B_2 = 180° - 48°26' = 131°34'$ leads to another solution. Thus,

$$B_1 = 48°26', \quad C_1 = 180° - (36°15' + 48°26') = 95°19',$$

and

$$B_2 = 131°34', \quad C_2 = 180° - (36°15' + 131°34') = 12°11'.$$

To find c_1, we use the law of sines again. Thus,

$$c_1 = \frac{9.8 \sin 95°19'}{\sin 36°15'} = 16.5.$$

Similarly, we have

$$c_2 = \frac{9.8 \sin 12°11'}{\sin 36°15'} = 3.5.$$

As a partial check, the equation $c_1 = b \cos A + a \cos B_1$ may be used. Thus,

$$12.4 \cos 36°15' + 9.8 \cos 48°26' = 12.4\ (0.8064) + 9.8\ (0.6635) = 16.5.$$

This result is the same as the value previously calculated. The same method may be applied to check c_2.

Example 18–3. Given $A = 56°30$, $a = 13.0$, $b = 10.7$, solve the triangle ABC.

Solution: Here we clearly have only one solution, since $a > b$. This can be seen geometrically if we draw Fig. 18–5 approximately to scale and show the given parts. Since a is greater than b, an arc with C as center intersects AX on opposite sides of A. Obviously there is only one triangle, AB_1C, containing the angle A.

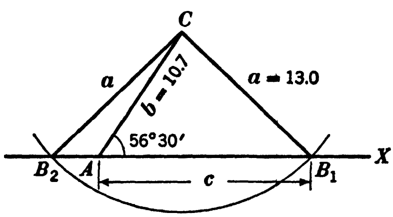

FIG. 18–5.

To find B_1, we use the relationship

$$\frac{\sin B_1}{10.7} = \frac{\sin 56°30'}{13},$$

whence

$$\log \sin B_1 = \log 10.7 + \log \sin 56°30' - \log 13.$$

This gives

$$B_1 = 43°20'.$$

Then $C = 180° - (56°30' + 43°20') = 80°10'$.

To find c, we use the law of sines and obtain

$$\frac{c}{\sin 80°10'} = \frac{13}{\sin 56°30'}.$$

This gives $c = 15.4$.

Example 18–4. Given $A = 67°40'$, $a = 16.0$, $b = 17.3$, solve the triangle.

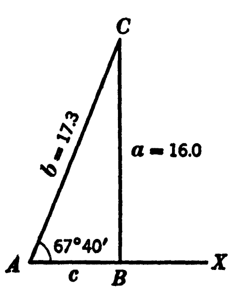

FIG. 18–6.

Solution: The given parts are shown in Fig. 18–6. By the law of sines, we have

$$\frac{\sin B}{17.3} = \frac{\sin 67°40'}{16}.$$

Therefore,

$$\log \sin B = \log 17.3 + \log \sin 67°40' - \log 16$$
$$= (1.2380) + (9.9661 - 10) - (1.2041) = 0.$$

Hence, $B = 90°$.

The solution may be completed by applying the theory of right triangles. Only one solution exists.

Example 18–5. Given $A = 47°23'$, $a = 230$, $b = 720$, solve the triangle.

Solution: From Fig. 18–7, it appears that no triangle is possible. The following work verifies this fact.

To find B, use the relationship

$$\frac{\sin B}{b} = \frac{\sin A}{a},$$

obtaining

$$\sin B = \frac{720 \sin 47°23'}{230}.$$

The logarithmic work follows:

$$\begin{aligned} \log 720 &= 2.8573 \\ \log \sin 47°23' &= \underline{9.8668 - 10} \\ & 12.7241 - 10 \\ \log 230 &= \underline{2.3617} \\ \log \sin B &= 10.3624 - 10 \\ &= 0.3624. \end{aligned}$$

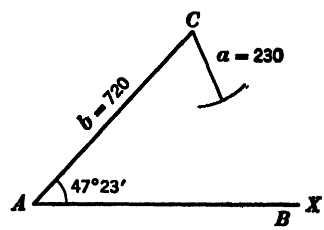

Fig. 18–7.

Since $\log \sin B > 0$, $\sin B$ would have to be greater than 1, which is impossible. Therefore, there is no solution.

EXERCISE 18–1

In each of the problems from 1 to 12, solve the given triangle by the law of sines.

1. $a = 12.30$, $A = 36°25'$, $B = 44°37'$.
2. $b = 12.18$, $A = 47°33'$, $B = 67°51'$.
3. $c = 461.3$, $B = 67°19'$, $C = 23°14'$.
4. $b = 0.6384$, $B = 39°39'$, $C = 87°16'$.
5. $a = 6.714$, $A = 37°53'$, $C = 136°36'$.
6. $c = 7832$, $A = 68°39'$, $B = 43°58'$.
7. $a = 21.23$, $c = 64.21$, $C = 62°31'$.
8. $b = 0.8146$, $c = 31.63$, $B = 11°19'$.
9. $a = 987.4$, $b = 503.6$, $A = 54°13'$.
10. $a = 0.003862$, $c = 0.0008157$, $A = 26°13'$.
11. $b = 1.386$, $c = 2.451$, $B = 83°19'$.
12. $b = 4.395$, $c = 9.806$, $C = 37°46'$.
13. A surveyor wishes to find the distance across a stream from point A to point B. He finds that the distance from A to a point C on the same side of the stream is 687.4 feet, and angles BAC and BCA are 49°53′ and 58°16′, respectively. Find the distance AB.
14. A surveyor was running a line due west when he reached a swamp. From the edge of the swamp he ran a line S 63° W for 2500 feet, and from this point he ran a line N 27°23′ W. How far had he gone on this line when he reached his original line produced? How far was it across the swamp?
15. A building 63.7 feet high stands on the top of a hill. From a point at the foot of the hill the angles of elevation to the top and bottom of the building are 42°16′ and 38°31′, respectively. Find the height of the hill.

16. From a certain point on the ground the angle of elevation to the top of a building is 46°17′. From a point on the ground 83 feet nearer the building the angle of elevation is 68°23′. Assuming that the ground is level, find the height of the building.
17. One side and a diagonal of a parallelogram are 14.63 inches and 21.4 inches, respectively. The angle between the diagonals and opposite the given side is 116°23′. Find the length of the other diagonal.
18. It is necessary to measure the distance between two artillery pieces A and B. The angle of depression from an observation point C to gun A is 24°47′. Sound travels at the rate of 1140 feet per second, and the sounds from guns A and B reach C in 2.3 and 1.7 seconds, respectively. Find the distance AB, assuming that points A, B, and C lie in the same plane.
19. A body is acted on by two forces, $F_1 = 2643$ pounds and $F_2 = 2341$ pounds. The resultant F_3 lies on a line making an angle of 46°33′ with F_1. Find F_3 and the angle between the lines of action of F_1 and F_2. (The resultant of two forces is their vector sum.)
20. A buoy, located at a point B, is 6 miles from a point A at one end of an island and 10 miles from a point C at the other end of the island. If the angle BAC is 132°16′, find the distance between the points A and C on the island.

18–5. THE LAW OF COSINES

Law of Cosines. Let ABC be any triangle. Then

$$a^2 = b^2 + c^2 - 2bc \cos A, \tag{18–8}$$

$$b^2 = a^2 + c^2 - 2ac \cos B, \tag{18–9}$$

$$c^2 = a^2 + b^2 - 2ab \cos C. \tag{18–10}$$

These relationships constitute the *law of cosines.*

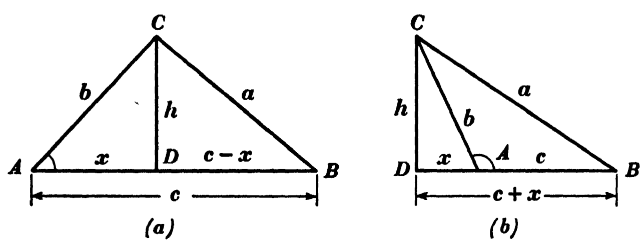

FIG. 18–8.

Proof. We shall establish (18–8) by considering the case when A is acute, as in Fig. 18–8(a) and the case when A is obtuse as in Fig. 18–8(b). The case $A = 90°$ involves no difficulty, and it will therefore be omitted.

Let h denote the altitude from C to the side AB. Also, let x denote the length AD. Hence, DB is $c - x$ in Fig. 18–8(a) and is $c + x$ in Fig. 18–8(b).

In Fig. 18–8(*a*),

$$(c - x)^2 + h^2 = a^2,$$

and

$$x^2 + h^2 = b^2.$$

Subtracting, we have $c^2 - 2cx = a^2 - b^2$, or

$$a^2 = b^2 + c^2 - 2cx.$$

Since $x = b \cos A$,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

In Fig. 18–8(*b*),

$$(c + x)^2 + h^2 = a^2,$$

and

$$x^2 + h^2 = b^2.$$

Subtracting, we find that $c^2 + 2cx = a^2 - b^2$, or

$$a^2 = b^2 + c^2 + 2cx.$$

Since $x = - b \cos A$ in this case,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

By drawing altitudes to the other sides and proceeding in a similar manner, we obtain (18–9) and (18–10).

Note. For a simple algebraic proof of the law of cosines, see problem 22 in Exercise 18–4. The law of cosines applies equally well if ABC is a right triangle. In this case, one of the formulas reduces to the pythogorean theorem, since $\cos 90° = 0$.

18–6. SOLUTION OF CASE III AND CASE IV BY THE LAW OF COSINES

Since the law of cosines is expressed by formulas involving addition and subtraction, it is not well adapted to logarithmic computation and its use is not recommended unless the given sides are easily squared.

Example 18–6. Given $b = 9.0$, $c = 13.0$, $A = 115°10'$, solve the triangle ABC.

Solution: By the law of cosines,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ &= 9^2 + (13)^2 - 2(9)(13) \cos 115°10' \\ &= 81 + 169 - 234(- \cos 64°50') \\ &= 250 + 234\,(0.4253) = 349.52. \end{aligned}$$

Hence,

$$a = 18.7.$$

We employ the law of sines to find angle B. Thus,

$$\frac{\sin B}{9} = \frac{\sin 115°10'}{18.7} = \frac{\sin 64°50'}{18.7}$$

and

$$B = 25°50'.$$

Therefore, $C = 180° - (115°10' + 25°50') = 39°$.

Example 18–7. Given $a = 3$, $b = 5$, $c = 7$. Find the angles.

Solution: From $a^2 = b^2 + c^2 - 2bc \cos A$, we have the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Therefore,

$$\cos A = \frac{25 + 49 - 9}{2(5)(7)} = 0.9286.$$

Hence,

$$A = 21°47'.$$

Similarly, $B = 38°13'$ and $C = 120°$.

We can check these by the equation $A + B + C = 180°$.

EXERCISE 18–2

In each of the problems from 1 to 6, solve the given triangle by the law of cosines.

1. $a = 300$, $b = 250$, $C = 58°40'$.
2. $a = 50$, $c = 240$, $B = 110°50'$.
3. $b = 65$, $c = 310$, $A = 67°10'$.
4. $a = 130$, $c = 90$, $B = 100°20'$.
5. $b = 50$, $c = 110$, $A = 150°$.
6. $a = 1.63$, $b = 3.45$, $C = 26°10'$.
7. If $a = 15$, $b = 12$, $c = 20$, find A.
8. If $a = 25$, $b = 30$, and $c = 35$, find B.
9. If $a = 100$, $b = 300$, $c = 500$, find C.
10. If $a = 15$, $b = 12$, and $c = 20$, find B.
11. If $a = 16$, $b = 17$, and $c = 18$, find A, B, C.
12. If $a = 260$, $b = 322$, $c = 481$, find A, B, C.
13. The distance between two points A and B cannot be measured directly. Accordingly, a third point C is selected, and it is found that $AC = 3000$ feet, $BC = 4500$ feet, and angle $ACB = 46°20'$. Find the distance AB.
14. Two sides of a parallelogram are 125 feet and 200 feet, and the included angle is $110°30'$. Find the length of the longer diagonal of the parallelogram and also the angle between that diagonal and a longer side of the parallelogram.
15. Two sides of a triangular plot of ground are 250 feet and 200 feet, and the included angle is $67°33'$. Find the perimeter of the plot.
16. Two sides of a parallelogram are 700 feet and 420 feet, and one diagonal is 600 feet. Find the length of the other diagonal.
17. In a triangle ABC, $a = 25$, $b = 27$, and the median from A is 20. Find c, A, B, C.

18–7. THE LAW OF TANGENTS

Law of Tangents. Let ABC be any triangle. Then the following relationships exist between two sides and the angles opposite them:

$$\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}, \tag{18–11}$$

$$(18\text{–}12) \qquad \frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)},$$

$$(18\text{–}13) \qquad \frac{c-a}{c+a} = \frac{\tan \frac{1}{2}(C-A)}{\tan \frac{1}{2}(C+A)}.$$

These relationships constitute the *law of tangents.*

Proof. Let us denote the common ratio of the law of sines by r. Thus, $a = r \sin A$, $b = r \sin B$, $c = r \sin C$. Then

$$\frac{a-b}{a+b} = \frac{r \sin A - r \sin B}{r \sin A + r \sin B} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

Substituting from (7–27) and (7–28) of Section 7–4, we have

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}.$$

Hence,

$$(18\text{–}11) \qquad \frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

A similar procedure may be followed to prove (18–12) and (18–13).

We shall now use the law of tangents to solve a triangle in which two sides and the included angle are given. Note that this law is well adapted to logarithmic computation.

Example 18–8. Given $b = 249$, $c = 372$, $A = 56°22'$, solve the triangle ABC.

Solution: To find $C - B$, we use the formula

$$\tan \frac{1}{2}(C-B) = \frac{c-b}{c+b} \tan \frac{1}{2}(C+B).$$

Here $c - b = 123$, $c + b = 621$, $C + B = 180° - A = 123°38'$, $\frac{1}{2}(C+B) = 61°49'$. Therefore,

$$\tan \frac{1}{2}(C-B) = \frac{123}{621} \tan 61°49',$$

and

$$\log \tan \frac{1}{2}(C-B) = \log 123 + \log \tan 61°49' - \log 621.$$

The logarithmic work follows:

$$\begin{aligned}
\log 123 &= 2.0899 \\
\log \tan 61°49' &= \underline{0.2710} \\
& 12.3609 - 10 \\
\log 621 &= \underline{2.7931} \\
\log \tan \frac{1}{2}(C - B) &= 9.5678 - 10 \\
\frac{1}{2}(C - B) &= 20°17'.
\end{aligned}$$

Hence,

$$C = \frac{1}{2}(C + B) + \frac{1}{2}(C - B) = 82°6',$$

and

$$B = \frac{1}{2}(C + B) - \frac{1}{2}(C - B) = 41°32'.$$

Check:

$$A + B + C = 56°22' + 82°6' + 41°32' = 180°.$$

To find a, we use the law of sines. Thus,

$$a = \frac{249 \sin 56°22'}{\sin 41°32'} = 313.$$

18–8. THE HALF-ANGLE FORMULAS

The following relationships are very convenient for the logarithmic solution of Case IV, where the three sides a, b, and c are known:

$$\text{(18–14)} \qquad \tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}},$$

$$\text{(18–15)} \qquad \tan \frac{B}{2} = \sqrt{\frac{(s - a)(s - c)}{s(s - b)}},$$

$$\text{(18–16)} \qquad \tan \frac{C}{2} = \sqrt{\frac{(s - a)(s - b)}{s(s - c)}}.$$

In these formulas, $s = \frac{1}{2}(a + b + c)$.

Proof. From (7–16) in Section 7–3, we have

$$\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}.$$

Also, from the law of cosines,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Therefore,

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc},$$

and

$$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c + a)(b + c - a)}{2bc}.$$

If we let $a + b + c = 2s$, then

$$a + b - c = a + b + c - 2c = 2(s - c),$$
$$a - b + c = a + b + c - 2b = 2(s - b),$$
$$b + c - a = a + b + c - 2a = 2(s - a).$$

Therefore,

$$\tan^2 \frac{A}{2} = \frac{(a + b - c)(a - b + c)}{(b + c + a)(b + c - a)} = \frac{(s - c)(s - b)}{s(s - a)}$$

and

$$\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}. \tag{18-14}$$

We can derive (18-15) and (18-16) in a similar manner.

Example 18-9. Given: $a = 379, b = 227, c = 416$, find the angles of the triangle.

Solution: Here $2s = a + b + c = 1022$. Then

$$s = 511,$$
$$s - a = 132,$$
$$s - b = 284,$$
$$s - c = 95.$$

Hence,

$$\tan \frac{A}{2} = \sqrt{\frac{(284)(95)}{(511)(132)}}.$$

The calculations by logarithms follow:

$$\begin{aligned} \log 284 &= 2.4533 \\ \log 95 &= \underline{1.9777} \\ \log (284)(95) &= 24.4310 - 20 \\ \log (511)(132) &= 4.8290 \\ 2 \,&\big|\, \underline{19.6020 - 20} \\ \log \tan \frac{A}{2} &= 9.8010 - 10 \\ A &= 64°38'. \end{aligned} \qquad \begin{aligned} \log 511 &= 2.7084 \\ \log 132 &= \underline{2.1206} \\ &\ 4.8290 \end{aligned}$$

Similarly,

$$\tan \frac{B}{2} = \sqrt{\frac{(132)(95)}{(511)(284)}},$$

$$\tan \frac{C}{2} = \sqrt{\frac{(132)(284)}{(511)(95)}}.$$

Hence, we find that $B = 32°46'$ and $C = 82°36'$.

Check:

$$A + B + C = 64°38' + 32°46' + 82°36' = 180°.$$

EXERCISE 18-3

In each of the problems from 1 to 10, solve the given triangle by the law of tangents if an angle is given, or by the half-angle formulas if three sides are given

1. $a = 50, b = 60, C = 60°$.
2. $b = 17.1, c = 22.3, A = 21°16'$.
3. $a = 230, c = 106, B = 95°10'$.
4. $b = 79.3, c = 113, A = 133°14'$.

5. $b = 41.82$, $c = 75.89$, $A = 78°49'$.
6. $a = 0.1028$, $b = 0.8726$, $C = 148°13'$.
7. $a = 625$, $b = 725$, $c = 825$.
8. $a = 60.65$, $b = 38.64$, $c = 23.57$.
9. $a = 67450$, $b = 84380$, $c = 98630$.
10. $a = 0.1146$, $b = 0.3184$, $c = 0.6379$.
11. The diagonals of a parallelogram are 6 inches and 10 inches, and they intersect at an angle of 63°. Find the sides of the parallelogram.
12. Points A and B are separated by an obstacle. In order to find the distance between them, a third point C is selected and it is found that $AC = 126$ rods and $BC = 185$ rods. The angle subtended at C by AB is 96°14′. Find AB.
13. Two circles whose radii are 14 and 17 inches respectively intersect. The angle between the tangents to the circles at either point of intersection is 38°46′. Find the distance between the centers of the circles.
14. The sides of a parallelogram are 13.4 inches and 18.5 inches, and one diagonal is 15.6 inches. Find the angles and the other diagonal of the parallelogram.
15. Three circles whose radii are 10, 11, and 12 inches, respectively, are tangent to each other externally. Find the angles of the triangle formed by joining their centers.
16. The sides of a triangular field are in the proportion 4:5:6. The area of the field is 18 acres. If there are 160 square rods in an acre, find the length of each side of the field in rods.
17. In triangle ABC, prove the following:

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C}$$

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2} C}$$

These formulas are called Mollweide's equations. They may be used in checking the solution of a triangle.

18–9. AREA OF A TRIANGLE

We can readily see that the area K of the triangle in Fig. 18–1 is

$$K = \frac{1}{2}ch = \frac{1}{2}cb \sin A. \tag{18–17}$$

In either triangle, $h = b \sin A$. In like manner, we obtain

$$K = \frac{1}{2}ac \sin B, \tag{18–18}$$

$$K = \frac{1}{2}ab \sin C. \tag{18–19}$$

By substituting $\dfrac{c \sin B}{\sin C}$ for b from the law of sines, we may transform (18–17) to obtain

$$K = \frac{c^2 \sin A \sin B}{2 \sin C} \tag{18–20}$$

By cyclic interchanges of letters, we obtain

$$K = \frac{a^2 \sin B \sin C}{2 \sin A}, \tag{18–21}$$

$$K = \frac{b^2 \sin A \sin C}{2 \sin B}. \tag{18–22}$$

To derive a formula for finding the area of a triangle when its three sides are given, we first transform (18–17) in the following manner:

$$\begin{aligned} K &= \frac{1}{2}bc \sin A = \frac{1}{2}bc \sin\left(2 \cdot \frac{A}{2}\right) \\ &= \frac{1}{2}bc\left(2 \sin \frac{A}{2} \cos \frac{A}{2}\right) \\ &= bc \sin \frac{A}{2} \cos \frac{A}{2}. \end{aligned}$$

But, from (7–14) and (7–15) in Section 7–3,

$$\sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} \quad \text{and} \quad \cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}$$

Using the values from Section 18–8 for $1 - \cos A$ and $1 + \cos A$, we have

$$\sin \frac{A}{2} = \sqrt{\frac{(s - c)(s - b)}{bc}} \quad \text{and} \quad \cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$$

Consequently, the formula for area in terms of the sides is

$$\begin{aligned} K &= bc\sqrt{\frac{(s - c)(s - b)}{bc}}\sqrt{\frac{s(s - a)}{bc}} \\ &= \sqrt{s(s - a)(s - b)(s - c)}. \end{aligned} \tag{18–21}$$

The following examples illustrate the use of the area formulas.

Example 18–10. Find the area of the triangle in Example 18–8, in which $b = 249$, $c = 372$, $A = 56°22'$.

Solution: Since two sides and the included angle are given, (18–17) may be used. Thus,

$$K = \frac{1}{2}bc \sin A = \frac{1}{2}(249)(372) \sin 56°22'.$$

The logarithmic work follows:

$$\begin{aligned} \log 0.5 &= 9.6990 - 10 \\ \log 249 &= 2.3962 \\ \log 372 &= 2.5705 \\ \log \sin 56°22' &= \underline{9.9205 - 10} \\ \log K &= 24.5862 - 20 \\ K &= 38{,}600. \end{aligned}$$

Example 18–11. Find the area of the triangle in Example 18–1, in which $A = 38°14'$, $B = 67°20'$, $c = 329$.

Solution: Since two angles and a side are given, (18–20) may be used. In this case,

$$K = \frac{c^2 \sin A \sin B}{2 \sin C} = \frac{(329)^2 \sin 38°14' \sin 67°20'}{2 \sin 74°26'} = 32{,}100.$$

Example 18–12. Find the area of the triangle in Example 18–9, in which $a = 379$, $b = 227$, $c = 416$.

Solution: In this solution, (18–21) is used. We have

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(511)(132)(284)(95)} = 42{,}700.$$

EXERCISE 18–4

In each of the problems from 1 to 8, find the area of the given triangle.

1. $a = 12.30$, $A = 36°25'$, $B = 44°37'$.
2. $c = 461.3$, $B = 67°19'$, $C = 23°14'$.
3. $a = 987.4$, $b = 503.6$, $A = 54°13'$.
4. $b = 4.395$, $c = 9.806$, $C = 37°46'$.
5. $b = 65$, $c = 310$, $A = 67°10'$.
6. $a = 300$, $b = 250$, $C = 58°40'$.
7. $a = 15$, $b = 12$, $c = 20$.
8. $a = 100$, $b = 300$, $c = 500$.
9. In triangle ABC, let r be the radius of the inscribed circle. Prove that $K = rs$ and, therefore, that $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.
10. Find the radius of the circle inscribed in the triangle whose sides are 48.92 feet, 63.86 feet, and 72.31 feet.
11. A cylindrical tank is to be built on a triangular lot having sides whose lengths are 200 feet, 186 feet, and 176 feet. Find the radius of the largest such tank which can be built on the lot.
12. In triangle ABC, let R be the radius of the circumscribed circle. Show that
$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
13. In triangle ABC, show that $R = \dfrac{abc}{4K}$, where R is the radius of the circumscribed circle and K is the area of the triangle.
14. The sides of a triangle are 23, 29, and 46 feet. Find the areas of the triangle and the inscribed and circumscribed circles.
15. The sides of a triangular plot of grass are 42 feet, 65 feet, and 87 feet. Find the minimum radius of action of an automatic lawn sprinkler which will water all parts of the plot from the same point.
16. An arc of a circle of radius r subtends a central angle θ. Show that the area bounded by this arc and its chord is $\dfrac{1}{2} r^2 (\theta - \sin \theta)$.
17. Find the area of the largest pentagon which can be cut from a circular piece of metal 4 feet in radius. How much metal is wasted?
18. In triangle ABC, prove that the median from any vertex to the side opposite divides the angle at that vertex into two parts whose sines are proportional to the lengths of the parts into which the side opposite is divided by the median.

19. In triangle ABC, prove that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}.$$

20. In triangle ABC, prove that

$$a + b + c = (b + c) \cos A + (c + a) \cos B + (a + b) \cos C.$$

21. In triangle ABC, prove that

$$a^2 + b^2 + c^2 = a^2(\cos^2 C + \sin^2 B) + b^2(\cos^2 A + \sin^2 C) + c^2(\cos^2 B + \sin^2 A).$$

22. In triangle ABC, show that

$$a = b \cos C + c \cos B,$$
$$b = c \cos A + a \cos C,$$
$$c = a \cos B + b \cos A.$$

Multiply the first equation by a, the second by b, and the third by c, to give a second proof of the law of cosines; that is, prove (18–8) by showing that $b^2 + c^2 - a^2 = 2bc \cos A$. Similarly prove (18–9) and (18–10).

23. Consider any triangle ABC. If $a > b$, prove that $A > B$. If $A > B$, prove that $a > b$.

24. In triangles ABC and $A'B'C'$, let A and A', B and B', C and C' be pairs of corresponding vertices, and let the corresponding sides be a and a', b and b', c and c'. If $a = a'$, $b = b'$, and $C > C'$, prove that $c > c'$. If $a = a'$, $b = b'$, and $c > c'$, prove that $C > C'$.

Appendix A Tables

TABLE I

Four-Place Values of Functions of Numbers

t	sin t	cos t	tan t	cot t	sec t	csc t
.00	.0000	1.0000	.0000	.....	1.000	.....
.01	.0100	1.0000	.0100	99.997	1.000	100.00
.02	.0200	.9998	.0200	49.993	1.000	50.00
.03	.0300	.9996	.0300	33.323	1.000	33.34
.04	.0400	.9992	.0400	24.987	1.001	25.01
.05	.0500	.9988	.0500	19.983	1.001	20.01
.06	.0600	.9982	.0601	16.647	1.002	16.68
.07	.0699	.9976	.0701	14.262	1.002	14.30
.08	.0799	.9968	.0802	12.473	1.003	12.51
.09	.0899	.9960	.0902	11.081	1.004	11.13
.10	.0998	.9950	.1003	9.967	1.005	10.02
.11	.1098	.9940	.1104	9.054	1.006	9.109
.12	.1197	.9928	.1206	8.293	1.007	8.353
.13	.1296	.9916	.1307	7.649	1.009	7.714
.14	.1395	.9902	.1409	7.096	1.010	7.166
.15	.1494	.9888	.1511	6.617	1.011	6.692
.16	.1593	.9872	.1614	6.197	1.013	6.277
.17	.1692	.9856	.1717	5.826	1.015	5.911
.18	.1790	.9838	.1820	5.495	1.016	5.586
.19	.1889	.9820	.1923	5.200	1.018	5.295
.20	.1987	.9801	.2027	4.933	1.020	5.033
.21	.2085	.9780	.2131	4.692	1.022	4.797
.22	.2182	.9759	.2236	4.472	1.025	4.582
.23	.2280	.9737	.2341	4.271	1.027	4.386
.24	.2377	.9713	.2447	4.086	1.030	4.207
.25	.2474	.9689	.2553	3.916	1.032	4.042
.26	.2571	.9664	.2660	3.759	1.035	3.890
.27	.2667	.9638	.2768	3.613	1.038	3.749
.28	.2764	.9611	.2876	3.478	1.041	3.619
.29	.2860	.9582	.2984	3.351	1.044	3.497
.30	.2955	.9553	.3093	3.233	1.047	3.384
.31	.3051	.9523	.3203	3.122	1.050	3.278
.32	.3146	.9492	.3314	3.018	1.053	3.179
.33	.3240	.9460	.3425	2.920	1.057	3.086
.34	.3335	.9428	.3537	2.827	1.061	2.999
.35	.3429	.9394	.3650	2.740	1.065	2.916
.36	.3523	.9359	.3764	2.657	1.068	2.839
.37	.3616	.9323	.3879	2.578	1.073	2.765
.38	.3709	.9287	.3994	2.504	1.077	2.696
.39	.3802	.9249	.4111	2.433	1.081	2.630
t	sin t	cos t	tan t	cot t	sec t	csc t

TABLE I (*continued*)

t	sin t	cos t	tan t	cot t	sec t	csc t
.40	.3894	.9211	.4228	2.365	1.086	2.568
.41	.3986	.9171	.4346	2.301	1.090	2.509
.42	.4078	.9131	.4466	2.239	1.095	2.452
.43	.4169	.9090	.4586	2.180	1.100	2.399
.44	.4259	.9048	.4708	2.124	1.105	2.348
.45	.4350	.9004	.4831	2.070	1.111	2.299
.46	.4439	.8961	.4954	2.018	1.116	2.253
.47	.4529	.8916	.5080	1.969	1.122	2.208
.48	.4618	.8870	.5206	1.921	1.127	2.166
.49	.4706	.8823	.5334	1.875	1.133	2.125
.50	.4794	.8776	.5463	1.830	1.139	2.086
.51	.4882	.8727	.5594	1.788	1.146	2.048
.52	.4969	.8678	.5726	1.747	1.152	2.013
.53	.5055	.8628	.5859	1.707	1.159	1.978
.54	.5141	.8577	.5994	1.668	1.166	1.945
.55	.5227	.8525	.6131	1.631	1.173	1.913
.56	.5312	.8473	.6269	1.595	1.180	1.883
.57	.5396	.8419	.6410	1.560	1.188	1.853
.58	.5480	.8365	.6552	1.526	1.196	1.825
.59	.5564	.8309	.6696	1.494	1.203	1.797
.60	.5646	.8253	.6841	1.462	1.212	1.771
.61	.5729	.8196	.6989	1.431	1.220	1.746
.62	.5810	.8139	.7139	1.401	1.229	1.721
.63	.5891	.8080	.7291	1.372	1.238	1.697
.64	.5972	.8021	.7445	1.343	1.247	1.674
.65	.6052	.7961	.7602	1.315	1.256	1.652
.66	.6131	.7900	.7761	1.288	1.266	1.631
.67	.6210	.7838	.7923	1.262	1.276	1.610
.68	.6288	.7776	.8087	1.237	1.286	1.590
.69	.6365	.7712	.8253	1.212	1.297	1.571
.70	.6442	.7648	.8423	1.187	1.307	1.552
.71	.6518	.7584	.8595	1.163	1.319	1.534
.72	.6594	.7518	.8771	1.140	1.330	1.517
.73	.6669	.7452	.8949	1.117	1.342	1.500
.74	.6743	.7385	.9131	1.095	1.354	1.483
.75	.6816	.7317	.9316	1.073	1.367	1.467
.76	.6889	.7248	.9505	1.052	1.380	1.452
.77	.6961	.7179	.9697	1.031	1.393	1.437
.78	.7033	.7109	.9893	1.011	1.407	1.422
.79	.7104	.7038	1.009	.9908	1.421	1.408
t	sin t	cos t	tan t	cot t	sec t	csc t

TABLE I (*continued*)

t	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
.80	.7174	.6967	1.030	.9712	1.435	1.394
.81	.7243	.6895	1.050	.9520	1.450	1.381
.82	.7311	.6822	1.072	.9331	1.466	1.368
.83	.7379	.6749	1.093	.9146	1.482	1.355
.84	.7446	.6675	1.116	.8964	1.498	1.343
.85	.7513	.6600	1.138	.8785	1.515	1.331
.86	.7578	.6524	1.162	.8609	1.533	1.320
.87	.7643	.6448	1.185	.8437	1.551	1.308
.88	.7707	.6372	1.210	.8267	1.569	1.297
.89	.7771	.6294	1.235	.8100	1.589	1.287
.90	.7833	.6216	1.260	.7936	1.609	1.277
.91	.7895	.6137	1.286	.7774	1.629	1.267
.92	.7956	.6058	1.313	.7615	1.651	1.257
.93	.8016	.5978	1.341	.7458	1.673	1.247
.94	.8076	.5898	1.369	.7303	1.696	1.238
.95	.8134	.5817	1.398	.7151	1.719	1.229
.96	.8192	.5735	1.428	.7001	1.744	1.221
.97	.8249	.5653	1.459	.6853	1.769	1.212
.98	.8305	.5570	1.491	.6707	1.795	1.204
.99	.8360	.5487	1.524	.6563	1.823	1.196
1.00	.8415	.5403	1.557	.6421	1.851	1.188
1.01	.8468	.5319	1.592	.6281	1.880	1.181
1.02	.8521	.5234	1.628	.6142	1.911	1.174
1.03	.8573	.5148	1.665	.6005	1.942	1.166
1.04	.8624	.5062	1.704	.5870	1.975	1.160
1.05	.8674	.4976	1.743	.5736	2.010	1.153
1.06	.8724	.4889	1.784	.5604	2.046	1.146
1.07	.8772	.4801	1.827	.5473	2.083	1.140
1.08	.8820	.4713	1.871	.5344	2.122	1.134
1.09	.8866	.4625	1.917	.5216	2.162	1.128
1.10	.8912	.4536	1.965	.5090	2.205	1.122
1.11	.8957	.4447	2.014	.4964	2.249	1.116
1.12	.9001	.4357	2.066	.4840	2.295	1.111
1.13	.9044	.4267	2.120	.4718	2.344	1.106
1.14	.9086	.4176	2.176	.4596	2.395	1.101
1.15	.9128	.4085	2.234	.4475	2.448	1.096
1.16	.9168	.3993	2.296	.4356	2.504	1.091
1.17	.9208	.3902	2.360	.4237	2.563	1.086
1.18	.9246	.3809	2.427	.4120	2.625	1.082
1.19	.9284	.3717	2.498	.4003	2.691	1.077
t	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$

TABLE I (*continued*)

t	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
1.20	.9320	.3624	2.572	.3888	2.760	1.073
1.21	.9356	.3530	2.650	.3773	2.833	1.069
1.22	.9391	.3436	2.733	.3659	2.910	1.065
1.23	.9425	.3342	2.820	.3546	2.992	1.061
1.24	.9458	.3248	2.912	.3434	3.079	1.057
1.25	.9490	.3153	3.010	.3323	3.171	1.054
1.26	.9521	.3058	3.113	.3212	3.270	1.050
1.27	.9551	.2963	3.224	.3102	3.375	1.047
1.28	.9580	.2867	2.341	.2993	3.488	1.044
1.29	.9608	.2771	3.467	.2884	3.609	1.041
1.30	.9636	.2675	3.602	.2776	3.738	1.038
1.31	.9662	.2579	3.747	.2669	3.878	1.035
1.32	.9687	.2482	3.903	.2562	4.029	1.032
1.33	.9711	.2385	4.072	.2456	4.193	1.030
1.34	.9735	.2288	4.256	.2350	4.372	1.027
1.35	.9757	.2190	4.455	.2245	4.566	1.025
1.36	.9779	.2092	4.673	.2140	4.779	1.023
1.37	.9799	.1994	4.913	.2035	5.014	1.021
1.38	.9819	.1896	5.177	.1931	5.273	1.018
1.39	.9837	.1798	5.471	.1828	5.561	1.017
1.40	.9854	.1700	5.798	.1725	5.883	1.015
1.41	.9871	.1601	6.165	.1622	6.246	1.013
1.42	.9887	.1502	6.581	.1519	6.657	1.011
1.43	.9901	.1403	7.055	.1417	7.126	1.010
1.44	.9915	.1304	7.602	.1315	7.667	1.009
1.45	.9927	.1205	8.238	.1214	8.299	1.007
1.46	.9939	.1106	8.989	.1113	9.044	1.006
1.47	.9949	.1006	9.887	.1011	9.938	1.005
1.48	.9959	.0907	10.983	.0910	11.029	1.004
1.49	.9967	.0807	12.350	.0810	12.390	1.003
1.50	.9975	.0707	14.101	.0709	14.137	1.003
1.51	.9982	.0608	16.428	.0609	16.458	1.002
1.52	.9987	.0508	19.670	.0508	19.695	1.001
1.53	.9992	.0408	24.498	.0408	24.519	1.001
1.54	.9995	.0308	32.461	.0308	32.476	1.000
1.55	.9998	.0208	48.078	.0208	48.089	1.000
1.56	.9999	.0108	92.620	.0108	92.626	1.000
1.57	1.0000	.0008	1255.8	.0008	1255.8	1.000
1.58	1.0000	−.0092	−108.65	−.0092	−108.65	1.000
1.59	.9998	−.0192	−52.067	−.0192	−52.08	1.000
1.60	.9996	−.0292	−34.233	−.0292	−34.25	1.000
t	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$

TABLE II

FOUR-PLACE VALUES OF FUNCTIONS

→	Sin	Cos	Tan	Cot	Sec	Csc	
0° 00′	.0000	1.000	.0000	——	1.000	——	**90° 00′**
10′	029	000	029	343.8	000	343.8	89° 50′
20′	058	000	058	171.9	000	171.9	40′
30′	.0087	1.000	.0087	114.6	1.000	114.6	30′
40′	116	.9999	116	85.94	000	85.95	20′
0° 50′	145	999	145	68.75	000	68.76	10′
1° 00′	.0175	.9998	.0175	57.29	1.000	57.30	**89° 00′**
10′	204	998	204	49.10	000	49.11	88° 50′
20′	233	997	233	42.96	000	42.98	40′
30′	.0262	.9997	.0262	38.19	1.000	38.20	30′
40′	291	996	291	34.37	000	34.38	20′
1° 50′	320	995	320	31.24	001	31.26	10′
2° 00′	.0349	.9994	.0349	28.64	1.001	28.65	**88° 00′**
10′	378	993	378	26.43	001	26.45	87° 50′
20′	407	992	407	24.54	001	24.56	40′
30′	.0436	.9990	.0437	22.90	1.001	22.93	30′
40′	465	989	466	21.47	001	21.49	20′
2° 50′	494	988	495	20.21	001	20.23	10′
3° 00′	.0523	.9986	.0524	19.08	1.001	19.11	**87° 00′**
10′	552	985	553	18.07	002	18.10	86° 50′
20′	581	983	582	17.17	002	17.20	40′
30′	.0610	.9981	.0612	16.35	1.002	16.38	30′
40′	640	980	641	15.60	002	15.64	20′
3° 50′	669	978	670	14.92	002	14.96	10′
4° 00′	.0698	.9976	.0699	14.30	1.002	14.34	**86° 00′**
10′	727	974	729	13.73	003	13.76	85° 50′
20′	756	971	758	13.20	003	13.23	40′
30′	.0785	.9969	.0787	12.71	1.003	12.75	30′
40′	814	967	816	12.25	003	12.29	20′
4° 50′	843	964	846	11.83	004	11.87	10′
5° 00′	.0872	.9962	.0875	11.43	1.004	11.47	**85° 00′**
10′	901	959	904	11.06	004	11.10	84° 50′
20′	929	957	934	10.71	004	10.76	40′
30′	.0958	.9954	.0963	10.39	1.005	10.43	30′
40′	.0987	951	.0992	10.08	005	10.13	20′
5° 50′	.1016	948	.1022	9.788	005	9.839	10′
6° 00′	.1045	.9945	.1051	9.514	1.006	9.567	**84° 00′**
	Cos	Sin	Cot	Tan	Csc	Sec	←

TABLE II (*continued*)

→	Sin	Cos	Tan	Cot	Sec	Csc	
6° 00′	.1045	.9945	.1051	9.514	1.006	9.567	**84° 00′**
10′	074	942	080	255	006	309	83° 50′
20′	103	939	110	9.010	006	9.065	40′
30′	.1132	.9936	.1139	8.777	1.006	8.834	30′
40	161	932	169	556	007	614	20′
6° 50′	190	929	198	345	007	405	10′
7° 00′	.1219	.9925	.1228	8.144	1.008	8.206	**83° 00′**
10′	248	922	257	7.953	008	8.016	82° 50′
20′	276	918	287	770	008	7.834	40′
30′	.1305	.9914	.1317	7.596	1.009	7.661	30′
40′	334	911	346	429	009	496	20′
7° 50′	363	907	376	269	009	337	10′
8° 00′	.1392	.9903	.1405	7.115	1.010	7.185	**82° 00′**
10′	421	899	435	6.968	010	7.040	81° 50′
20′	449	894	465	827	011	6.900	40′
30′	.1478	.9890	.1495	6.691	1.011	6.765	30′
40′	507	886	524	561	012	636	20′
8° 50′	536	881	554	435	012	512	10′
9° 00′	.1564	.9877	.1584	6.314	1.012	6.392	**81° 00′**
10′	593	872	614	197	013	277	80° 50′
20′	622	868	644	6.084	013	166	40′
30′	.1650	.9863	.1673	5.976	1.014	6.059	30′
40′	679	858	703	871	014	5.955	20′
9° 50′	708	853	733	769	015	855	10′
10° 00′	.1736	.9848	.1763	5.671	1.015	5.759	**80° 00′**
10′	765	843	793	576	016	665	79° 50′
20′	794	838	823	485	016	575	40′
30′	.1822	.9833	.1853	5.396	1.017	5.487	30′
40′	851	827	883	309	018	403	20′
10° 50′	880	822	914	226	018	320	10′
11° 00′	.1908	.9816	.1944	5.145	1.019	5.241	**79° 00′**
10′	937	811	.1974	5.066	019	164	78° 50′
20′	965	805	.2004	4.989	020	089	40′
30′	.1994	.9799	.2035	4.915	1.020	5.016	30′
40′	.2022	793	065	843	021	4.945	20′
11° 50′	051	787	095	773	022	876	10′
12° 00′	.2079	.9781	.2126	4.705	1.022	4.810	**78° 00′**
	Cos	Sin	Cot	Tan	Csc	Sec	←

TABLE II (*continued*)

→	Sin	Cos	Tan	Cot	Sec	Csc	
12° 00′	.2079	.9781	.2126	4.705	1.022	4.810	**78° 00′**
10′	108	775	156	638	023	745	77° 50′
20′	136	769	186	574	024	682	40′
30′	.2164	.9763	.2217	4.511	1.024	4.620	30′
40′	193	757	247	449	025	560	20′
12° 50′	221	750	278	390	026	502	10′
13° 00′	.2250	.9744	.2309	4.331	1.026	4.445	**77° 00′**
10′	278	737	339	275	027	390	76° 50′
20′	306	730	370	219	028	336	40′
30′	.2334	.9724	.2401	4.165	1.028	4.284	30′
40′	363	717	432	113	029	232	20′
13° 50′	391	710	462	061	030	182	10′
14° 00′	.2419	.9703	.2493	4.011	1.031	4.134	**76° 00′**
10′	447	696	524	3.962	031	086	75° 50′
20′	476	689	555	914	032	4.039	40′
30′	.2504	.9681	.2586	3.867	1.033	3.994	30′
40′	532	674	617	821	034	950	20′
14° 50′	560	667	648	776	034	906	10′
15° 00′	.2588	.9659	.2679	3.732	1.035	3.864	**75° 00′**
10′	616	652	711	689	036	822	74° 50′
20′	644	644	742	647	037	782	40′
30′	.2672	.9636	.2773	3.606	1.038	3.742	30′
40′	700	628	805	566	039	703	20′
15° 50′	728	621	836	526	039	665	10′
16° 00′	.2756	.9613	.2867	3.487	1.040	3.628	**74° 00′**
10′	784	605	899	450	041	592	73° 50′
20′	812	596	931	412	042	556	40′
30′	.2840	.9588	.2962	3.376	1.043	3.521	30′
40′	868	580	.2994	340	044	487	20′
16° 50′	896	572	.3026	305	045	453	10′
17° 00′	.2924	.9563	.3057	3.271	1.046	3.420	**73° 00′**
10′	952	555	089	237	047	388	72° 50′
20′	.2979	546	121	204	048	356	40′
30′	.3007	.9537	.3153	3.172	1.049	3.326	30′
40′	035	528	185	140	049	295	20′
17° 50′	062	520	217	108	050	265	10′
18° 00′	.3090	.9511	.3249	3.078	1.051	3.236	**72° 00′**
	Cos	Sin	Cot	Tan	Csc	Sec	←

TABLE II (*continued*)

→	Sin	Cos	Tan	Cot	Sec	Csc	
18° 00′	.3090	.9511	.3249	3.078	1.051	3.236	**72° 00′**
10′	118	502	281	047	052	207	71° 50′
20′	145	492	314	3.018	053	179	40′
30′	.3173	.9483	.3346	2.989	1.054	3.152	30′
40′	201	474	378	960	056	124	20′
18° 50′	228	465	411	932	057	098	10′
19° 00′	.3256	.9455	.3443	2.904	1.058	3.072	**71° 00′**
10′	283	446	476	877	059	046	70° 50′
20′	311	436	508	850	060	3.021	40′
30′	.3338	.9426	.3541	2.824	1.061	2.996	30′
40′	365	417	574	798	062	971	20′
19° 50′	393	407	607	773	063	947	10′
20° 00′	.3420	.9397	.3640	2.747	1.064	2.924	**70° 00′**
10′	448	387	673	723	065	901	69° 50′
20′	475	377	706	699	066	878	40′
30′	.3502	.9367	.3739	2.675	1.068	2.855	30′
40′	529	356	772	651	069	833	20′
20° 50′	557	346	805	628	070	812	10′
21° 00′	.3584	.9336	.3839	2.605	1.071	2.790	**69° 00′**
10′	611	325	872	583	072	769	68° 50′
20′	638	315	906	560	074	749	40′
30′	.3665	.9304	.3939	2.539	1.075	2.729	30′
40′	692	293	.3973	517	076	709	20′
21° 50′	719	283	.4006	496	077	689	10′
22° 00′	.3746	.9272	.4040	2.475	1.079	2.669	**68° 00′**
10′	773	261	074	455	080	650	67° 50′
20′	800	250	108	434	081	632	40′
30′	.3827	.9239	.4142	2.414	1.082	2.613	30′
40′	854	228	176	394	084	595	20′
22° 50′	881	216	210	375	085	577	10′
23° 00′	.3907	.9205	.4245	2.356	1.086	2.559	**67° 00′**
10′	934	194	279	337	088	542	66° 50′
20′	961	182	314	318	089	525	40′
30′	.3987	.9171	.4348	2.300	1.090	2.508	30′
40′	.4014	159	383	282	092	491	20′
23° 50′	041	147	417	264	093	475	10′
24° 00′	.4067	.9135	.4452	2.246	1.095	2.459	**66° 00′**
	Cos	Sin	Cot	Tan	Csc	Sec	←

TABLE II (*continued*)

→	Sin	Cos	Tan	Cot	Sec	Csc	
24° 00′	.4067	.9135	.4452	2.246	1.095	2.459	**66° 00′**
10′	094	124	487	229	096	443	65° 50′
20′	120	112	522	211	097	427	40′
30′	.4147	.9100	.4557	2.194	1.099	2.411	30′
40′	173	088	592	177	100	396	20′
24° 50′	200	075	628	161	102	381	10′
25° 00′	.4226	.9063	.4663	2.145	1.103	2.366	**65° 00′**
10′	253	051	699	128	105	352	64° 50′
20′	279	038	734	112	106	337	40′
30′	.4305	.9026	.4770	2.097	1.108	2.323	30′
40′	331	013	806	081	109	309	20′
25° 50′	358	.9001	841	066	111	295	10′
26° 00′	.4384	.8988	.4877	2.050	1.113	2.281	**64° 00′**
10′	410	975	913	035	114	268	63° 50′
20′	436	962	950	020	116	254	40′
30′	.4462	.8949	.4986	2.006	1.117	2.241	30′
40′	488	936	.5022	1.991	119	228	20′
26° 50′	514	923	059	977	121	215	10′
27° 00′	.4540	.8910	.5095	1.963	1.122	2.203	**63° 00′**
10′	566	897	132	949	124	190	62° 50′
20′	592	884	169	935	126	178	40′
30′	.4617	.8870	.5206	1.921	1.127	2.166	30′
40′	643	857	243	907	129	154	20′
27° 50′	669	843	280	894	131	142	10′
28° 00′	.4695	.8829	.5317	1.881	1.133	2.130	**62° 00′**
10′	720	816	354	868	134	118	61° 50′
20′	746	802	392	855	136	107	40′
30′	.4772	.8788	.5430	1.842	1.138	2.096	30′
40′	797	774	467	829	140	085	20′
28° 50′	823	760	505	816	142	074	10′
29° 00′	.4848	.8746	.5543	1.804	1.143	2.063	**61° 00′**
10′	874	732	581	792	145	052	60° 50′
20′	899	718	619	780	147	041	40′
30′	.4924	.8704	.5658	1.767	1.149	2.031	30′
40′	950	689	696	756	151	020	20′
29° 50′	.4975	675	735	744	153	010	10′
30° 00′	.5000	.8660	.5774	1.732	1.155	2.000	**60° 00′**
	Cos	Sin	Cot	Tan	Csc	Sec	←

TABLE II (*continued*)

→	Sin	Cos	Tan	Cot	Sec	Csc	
30° 00′	.5000	.8660	.5774	1.732	1.155	2.000	**60° 00′**
10′	025	646	812	720	157	1.990	59° 50′
20′	050	631	851	709	159	980	40′
30′	.5075	.8616	.5890	1.698	1.161	1.970	30′
40′	100	601	930	686	163	961	20′
30° 50′	125	587	.5969	675	165	951	10′
31° 00′	.5150	.8572	.6009	1.664	1.167	1.942	**59° 00′**
10′	175	557	048	653	169	932	58° 50′
20′	200	542	088	643	171	923	40′
30′	.5225	.8526	.6128	1.632	1.173	1.914	30′
40′	250	511	168	621	175	905	20′
31° 50′	275	496	208	611	177	896	10′
32° 00′	.5299	.8480	.6249	1.600	1.179	1.887	**58° 00′**
10′	324	465	289	590	181	878	57° 50′
20′	348	450	330	580	184	870	40′
30′	.5373	.8434	.6371	1.570	1.186	1.861	30′
40′	398	418	412	560	188	853	20′
32° 50′	422	403	453	550	190	844	10′
33° 00′	.5446	.8387	.6494	1.540	1.192	1.836	**57° 00′**
10′	471	371	536	530	195	828	56° 50′
20′	495	355	577	520	197	820	40′
30′	.5519	.8339	.6619	1.511	1.199	1.812	30′
40′	544	323	661	501	202	804	20′
33° 50′	568	307	703	492	204	796	10′
34° 00′	.5592	.8290	.6745	1.483	1.206	1.788	**56° 00′**
10′	616	274	787	473	209	781	55° 50′
20′	640	258	830	464	211	773	40′
30	.5664	.8241	.6873	1.455	1.213	1.766	30′
40	688	225	916	446	216	758	20′
34° 50	712	208	.6959	437	218	751	10′
35° 00′	.5736	.8192	.7002	1.428	1.221	1.743	**55° 00′**
10′	760	175	046	419	223	736	54° 50′
20′	783	158	089	411	226	729	40′
30′	.5807	.8141	.7133	1.402	1.228	1.722	30′
40′	831	124	177	393	231	715	20′
35° 50′	854	107	221	385	233	708	10′
36° 00′	.5878	.8090	.7265	1.376	1.236	1.701	**54° 00′**
	Cos	Sin	Cot	Tan	Csc	Sec	←

TABLE II (*continued*)

→	Sin	Cos	Tan	Cot	Sec	Csc	
36° 00′	.5878	.8090	.7265	1.376	1.236	1.701	**54° 00′**
10′	901	073	310	368	239	695	53° 50′
20′	925	056	355	360	241	688	40′
30′	.5948	.8039	.7400	1.351	1.244	1.681	30′
40′	972	021	445	343	247	675	20′
36° 50′	.5995	.8004	490	335	249	668	10′
37° 00′	.6018	.7986	.7536	1.327	1.252	1.662	**53° 00′**
10′	041	969	581	319	255	655	52° 50′
20′	065	951	627	311	258	649	40′
30′	.6088	.7934	.7673	1.303	1.260	1.643	30′
40′	111	916	720	295	263	636	20′
37° 50′	134	898	766	288	266	630	10′
38° 00′	.6157	.7880	.7813	1.280	1.269	1.624	**52° 00′**
10′	180	862	860	272	272	618	51° 50′
20′	202	844	907	265	275	612	40′
30′	.6225	.7826	.7954	1.257	1.278	1.606	30′
40′	248	808	.8002	250	281	601	20′
38° 50′	271	790	050	242	284	595	10′
39° 00′	.6293	.7771	.8098	1.235	1.287	1.589	**51° 00′**
10′	316	753	146	228	290	583	50° 50′
20′	338	735	195	220	293	578	40′
30′	.6361	.7716	.8243	1.213	1.296	1.572	30′
40′	383	698	292	206	299	567	20′
39° 50′	406	679	342	199	302	561	10′
40° 00′	.6428	.7660	.8391	1.192	1.305	1.556	**50° 00**
10	450	642	441	185	309	550	49° 50′
20′	472	623	491	178	312	545	40′
30′	.6494	.7604	.8541	1.171	1.315	1.540	30′
40′	517	585	591	164	318	535	20′
40° 50′	539	566	642	157	322	529	10′
41° 00′	.6561	.7547	.8693	1.150	1.325	1.524	**49° 00′**
10′	583	528	744	144	328	519	48° 50′
20′	604	509	796	137	332	514	40′
30′	.6626	.7490	.8847	1.130	1.335	1.509	30′
40′	648	470	899	124	339	504	20′
41° 50′	670	451	.8952	117	342	499	10′
42° 00′	.6691	.7431	.9004	1.111	1.346	1.494	**48° 00′**
	Cos	Sin	Cot	Tan	Csc	Sec	←

TABLE II (*continued*)

→	Sin	Cos	Tan	Cot	Sec	Csc	
42° 00′	.6691	.7431	.9004	1.111	1.346	1.494	**48° 00′**
10′	713	412	057	104	349	490	47° 50′
20′	734	392	110	098	353	485	40′
30′	.6756	.7373	.9163	1.091	1.356	1.480	30′
40′	777	353	217	085	360	476	20′
42° 50′	799	333	271	079	364	471	10′
43° 00′	.6820	.7314	.9325	1.072	1.367	1.466	**47° 00′**
10′	841	294	380	066	371	462	46° 50′
20′	862	274	435	060	375	457	40′
30′	.6884	.7254	.9490	1.054	1.379	1.453	30′
40′	905	234	545	048	382	448	20′
43° 50′	926	214	601	042	386	444	10′
44° 00′	.6947	.7193	.9657	1.036	1.390	1.440	**46° 00′**
10′	967	173	713	030	394	435	45° 50′
20′	.6988	153	770	024	398	431	40′
30′	.7009	.7133	.9827	1.018	1.402	1.427	30′
40′	030	112	884	012	406	423	20′
44° 50′	050	092	.9942	006	410	418	10′
45° 00′	.7071	.7071	1.000	1.000	1.414	1.414	**45° 00′**
	Cos	**Sin**	**Cot**	**Tan**	**Csc**	**Sec**	←

TABLE III

Four-Place Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
10	.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	.5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	.5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	.5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	.5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

TABLE III (*continued*)

N	0	1	2	3	4	5	6	7	8	9
40	.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	.6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	.6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	.6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	.6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	.6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	.6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	.6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	.6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	.6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	.7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	.7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	.7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	.7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	.7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	.7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	.7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	.7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	.7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	.7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	.7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	.7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	.8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	.8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	.8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	.8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	.8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	.8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

TABLE III (*continued*)

N	0	1	2	3	4	5	6	7	8	9
70	.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	.8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	.8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	.8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	.8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	.8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	.8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	.8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	.8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	.8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	.9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	.9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	.9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	.9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	.9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	.9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	.9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	.9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	.9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	.9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	.9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	.9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	.9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	.9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	.9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	.9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	.9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	.9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	.9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

*

TABLE IV

Four-Place Logarithms of Functions

→	L Sin	L Tan	L Cot	L Cos	
0° 00′				10.0000	**90° 00′**
10′	7.4637	7.4637	12.5363	.0000	89° 50′
20′	.7648	.7648	.2352	.0000	40′
30′	7.9408	7.9409	12.0591	.0000	30′
40′	8.0658	8.0658	11.9342	.0000	20′
0° 50′	.1627	.1627	.8373	10.0000	10′
1° 00′	8.2419	8.2419	11.7581	9.9999	**89° 00′**
10′	.3088	.3089	.6911	.9999	88° 50′
20′	.3668	.3669	.6331	.9999	40′
30′	.4179	.4181	.5819	.9999	30′
40′	.4637	.4638	.5362	.9998	20′
1° 50′	.5050	.5053	.4947	.9998	10′
2° 00′	8.5428	8.5431	11.4569	9.9997	**88° 00′**
10′	.5776	.5779	.4221	.9997	87° 50′
20′	.6097	.6101	.3899	.9996	40′
30′	.6397	.6401	.3599	.9996	30′
40′	.6677	.6682	.3318	.9995	20′
2° 50′	.6940	.6945	.3055	.9995	10′
3° 00′	8.7188	8.7194	11.2806	9.9994	**87° 00′**
10′	.7423	.7429	.2571	.9993	86° 50′
20′	.7645	.7652	.2348	.9993	40′
30′	.7857	.7865	.2135	.9992	30′
40′	.8059	.8067	.1933	.9991	20′
3° 50′	.8251	.8261	.1739	.9990	10′
4° 00′	8.8436	8.8446	11.1554	9.9989	**86° 00′**
10′	.8613	.8624	.1376	.9989	85° 50′
20′	.8783	.8795	.1205	.9988	40′
30′	.8946	.8960	.1040	.9987	30′
40′	.9104	.9118	.0882	.9986	20′
4° 50′	.9256	.9272	.0728	.9985	10′
5° 00′	8.9403	8.9420	11.0580	9.9983	**85° 00′**
10′	.9545	.9563	.0437	.9982	84° 50′
20′	.9682	.9701	.0299	.9981	40′
30′	.9816	.9836	.0164	.9980	30′
40′	8.9945	8.9966	11.0034	.9979	20′
5° 50′	9.0070	9.0093	10.9907	.9977	10′
6° 00′	9.0192	9.0216	10.9784	9.9976	**84° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE IV (*continued*)

→	L Sin	L Tan	L Cot	L Cos	
6° 00′	9.0192	9.0216	10.9784	9.9976	**84° 00′**
10′	.0311	.0336	.9664	.9975	83° 50′
20′	.0426	.0453	.9547	.9973	40′
30′	.0539	.0567	.9433	.9972	30′
40′	.0648	.0678	.9322	.9971	20′
6° 50′	.0755	.0786	.9214	.9969	10′
7° 00′	9.0859	9.0891	10.9109	9.9968	**83° 00′**
10′	.0961	.0995	.9005	.9966	82° 50′
20′	.1060	.1096	.8904	.9964	40′
30′	.1157	.1194	.8806	.9963	30′
40′	.1252	.1291	.8709	.9961	20′
7° 50′	.1345	.1385	.8615	.9959	10′
8° 00′	9.1436	9.1478	10.8522	9.9958	**82° 00′**
10′	.1525	.1569	.8431	.9956	81° 50′
20′	.1612	.1658	.8342	.9954	40′
30′	.1697	.1745	.8255	.9952	30′
40′	.1781	.1831	.8169	.9950	20′
8° 50′	.1863	.1915	.8085	.9948	10′
9° 00′	9.1943	9.1997	10.8003	9.9946	**81° 00′**
10′	.2022	.2078	.7922	.9944	80° 50′
20′	.2100	.2158	.7842	.9942	40′
30′	.2176	.2236	.7764	.9940	30′
40′	.2251	.2313	.7687	.9938	20′
9° 50′	.2324	.2389	.7611	.9936	10′
10° 00′	9.2397	9.2463	10.7537	9.9934	**80° 00′**
10′	.2468	.2536	.7464	.9931	79° 50′
20′	.2538	.2609	.7391	.9929	40′
30′	.2606	.2680	.7320	.9927	30′
40′	.2674	.2750	.7250	.9924	20′
10° 50′	.2740	.2819	.7181	.9922	10′
11° 00′	9.2806	9.2887	10.7113	9.9919	**79° 00′**
10′	.2870	.2953	.7047	.9917	78° 50′
20′	.2934	.3020	.6980	.9914	40′
30′	.2997	.3085	.6915	.9912	30′
40′	.3058	.3149	.6851	.9909	20′
11° 50′	.3119	.3212	.6788	.9907	10′
12° 00′	9.3179	9.3275	10.6725	9.9904	**78° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE IV (*continued*)

→	L Sin	L Tan	L Cot	L Cos	
12° 00′	9.3179	9.3275	10.6725	9.9904	**78° 00′**
10′	.3238	.3336	.6664	.9901	77° 50′
20′	.3296	.3397	.6603	.9899	40′
30′	.3353	.3458	.6542	.9896	30′
40′	.3410	.3517	.6483	.9893	20′
12° 50′	.3466	.3576	.6424	.9890	10′
13° 00′	9.3521	9.3634	10.6366	9.9887	**77° 00′**
10′	.3575	.3691	.6309	.9884	76° 50′
20′	.3629	.3748	.6252	.9881	40′
30′	.3682	.3804	.6196	.9878	30′
40′	.3734	.3859	.6141	.9875	20′
13° 50′	.3786	.3914	.6086	.9872	10′
14° 00′	9.3837	9.3968	10.6032	9.9869	**76° 00′**
10′	.3887	.4021	.5979	.9866	75° 50′
20′	.3937	.4074	.5926	.9863	40′
30′	.3986	.4127	.5873	.9859	30′
40′	.4035	.4178	.5822	.9856	20′
14° 50′	.4083	.4230	.5770	.9853	10′
15° 00′	9.4130	9.4281	10.5719	9.9849	**75° 00′**
10′	.4177	.4331	.5669	.9846	74° 50′
20′	.4223	.4381	.5619	.9843	40′
30′	.4269	.4430	.5570	.9839	30′
40′	.4314	.4479	.5521	.9836	20′
15° 50′	.4359	.4527	.5473	.9832	10′
16° 00′	9.4403	9.4575	10.5425	9.9828	**74° 00′**
10′	.4447	.4622	.5378	.9825	73° 50′
20′	.4491	.4669	.5331	.9821	40′
30′	.4533	.4716	.5284	.9817	30′
40′	.4576	.4762	.5238	.9814	20′
16° 50′	.4618	.4808	.5192	.9810	10′
17° 00′	9.4659	9.4853	10.5147	9.9806	**73° 00′**
10′	.4700	.4898	.5102	.9802	72° 50′
20′	.4741	.4943	.5057	.9798	40′
30′	.4781	.4987	.5013	.9794	30′
40′	.4821	.5031	.4969	.9790	20′
17° 50′	.4861	.5075	.4925	.9786	10′
18° 00′	9.4900	9.5118	10.4882	9.9782	**72° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE IV (*continued*)

→	L Sin	L Tan	L Cot	L Cos	
18° 00′	9.4900	9.5118	10.4882	9.9782	**72° 00′**
10′	.4939	.5161	.4839	.9778	71° 50′
20′	.4977	.5203	.4797	.9774	40′
30′	.5015	.5245	.4755	.9770	30′
40′	.5052	.5287	.4713	.9765	20′
18° 50′	.5090	.5329	.4671	.9761	10′
19° 00′	9.5126	9.5370	10.4630	9.9757	**71° 00′**
10′	.5163	.5411	.4589	.9752	70° 50′
20′	.5199	.5451	.4549	.9748	40′
30′	.5235	.5491	.4509	.9743	30′
40′	.5270	.5531	.4469	.9739	20′
19° 50′	.5306	.5571	.4429	.9734	10′
20° 00′	9.5341	9.5611	10.4389	9.9730	**70° 00′**
10′	.5375	.5650	.4350	.9725	69° 50′
20′	.5409	.5689	.4311	.9721	40′
30′	.5443	.5727	.4273	.9716	30′
40′	.5477	.5766	.4234	.9711	20′
20° 50′	.5510	.5804	.4196	.9706	10′
21° 00′	9.5543	9.5842	10.4158	9.9702	**69° 00′**
10′	.5576	.5879	.4121	.9697	68° 50′
20′	.5609	.5917	.4083	.9692	40′
30′	.5641	.5954	.4046	.9687	30′
40′	.5673	.5991	.4009	.9682	20′
21° 50′	.5704	.6028	.3972	.9677	10′
22° 00′	9.5736	9.6064	10.3936	9.9672	**68° 00′**
10′	.5767	.6100	.3900	.9667	67° 50′
20′	.5798	.6136	.3864	.9661	40′
30′	.5828	.6172	.3828	.9656	30′
40′	.5859	.6208	.3792	.9651	20′
22° 50′	.5889	.6243	.3757	.9646	10′
23° 00′	9.5919	9.6279	10.3721	9.9640	**67° 00′**
10′	.5948	.6314	.3686	.9635	66° 50′
20′	.5978	.6348	.3652	.9629	40′
30′	.6007	.6383	.3617	.9624	30′
40′	.6036	.6417	.3583	.9618	20′
23° 50′	.6065	.6452	.3548	.9613	10′
24° 00′	9.6093	9.6486	10.3514	9.9607	**66° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE IV (*continued*)

→	L Sin	L Tan	L Cot	L Cos	
24° 00′	9.6093	9.6486	10.3514	9.9607	**66° 00′**
10′	.6121	.6520	.3480	.9602	65° 50′
20′	.6149	.6553	.3447	.9596	40′
30′	.6177	.6587	.3413	.9590	30′
40′	.6205	.6620	.3380	.9584	20′
24° 50′	.6232	.6654	.3346	.9579	10′
25° 00′	9.6259	9.6687	10.3313	9.9573	**65° 00′**
10′	.6286	.6720	.3280	.9567	64° 50′
20′	.6313	.6752	.3248	.9561	40′
30′	.6340	.6785	.3215	.9555	30′
40′	.6366	.6817	.3183	.9549	20′
25° 50′	.6392	.6850	.3150	.9543	10′
26° 00′	9.6418	9.6882	10.3118	9.9537	**64° 00′**
10′	.6444	.6914	.3086	.9530	63° 50′
20′	.6470	.6946	.3054	.9524	40′
30′	.6495	.6977	.3023	.9518	30′
40′	.6521	.7009	.2991	.9512	20′
26° 50′	.6546	.7040	.2960	.9505	10′
27° 00′	9.6570	9.7072	10.2928	9.9499	**63° 00′**
10′	.6595	.7103	.2897	.9492	62° 50′
20′	.6620	.7134	.2866	.9486	40′
30′	.6644	.7165	.2835	.9479	30′
40′	.6668	.7196	.2804	.9473	20′
27° 50′	.6692	.7226	.2774	.9466	10′
28° 00′	9.6716	9.7257	10.2743	9.9459	**62° 00′**
10′	.6740	.7287	.2713	.9453	61° 50′
20′	.6763	.7317	.2683	.9446	40′
30′	.6787	.7348	.2652	.9439	30′
40′	.6810	.7378	.2622	.9432	20′
28° 50′	.6833	.7408	.2592	.9425	10′
29° 00′	9.6856	9.7438	10.2562	9.9418	**61° 00′**
10′	.6878	.7467	.2533	.9411	60° 50′
20′	.6901	.7497	.2503	.9404	40′
30′	.6923	.7526	.2474	.9397	30′
40′	.6946	.7556	.2444	.9390	20′
29° 50′	.6968	.7585	.2415	.9383	10′
30° 00′	9.6990	9.7614	10.2386	9.9375	**60° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE IV (*continued*)

→	L Sin	L Tan	L Cot	L Cos	
30° 00′	9.6990	9.7614	10.2386	9.9375	**60° 00′**
10′	.7012	.7644	.2356	.9368	59° 50′
20′	.7033	.7673	.2327	.9361	40′
30′	.7055	.7701	.2299	.9353	30′
40′	.7076	.7730	.2270	.9346	20′
30° 50′	.7097	.7759	.2241	.9338	10′
31° 00′	9.7118	9.7788	10.2212	9.9331	**59° 00′**
10′	.7139	.7816	.2184	.9323	58° 50′
20′	.7160	.7845	.2155	.9315	40′
30′	.7181	.7873	.2127	.9308	30′
40′	.7201	.7902	.2098	.9300	20′
31° 50′	.7222	.7930	.2070	.9292	10′
32° 00′	9.7242	9.7958	10.2042	9.9284	**58° 00′**
10′	.7262	.7986	.2014	.9276	57° 50′
20′	.7282	.8014	.1986	.9268	40′
30′	.7302	.8042	.1958	.9260	30′
40′	.7322	.8070	.1930	.9252	20′
32° 50′	.7342	.8097	.1903	.9244	10′
33° 00′	9.7361	9.8125	10.1875	9.9236	**57° 00′**
10′	.7380	.8153	.1847	.9228	56° 50′
20′	.7400	.8180	.1820	.9219	40′
30′	.7419	.8208	.1792	.9211	30′
40′	.7438	.8235	.1765	.9203	20′
33° 50′	.7457	.8263	.1737	.9194	10′
34° 00′	9.7476	9.8290	10.1710	9.9186	**56° 00′**
10′	.7494	.8317	.1683	.9177	55° 50′
20′	.7513	.8344	.1656	.9169	40′
30′	.7531	.8371	.1629	.9160	30′
40′	.7550	.8398	.1602	.9151	20′
34° 50′	.7568	.8425	.1575	.9142	10′
35° 00′	9.7586	9.8452	10.1548	9.9134	**55° 00′**
10′	.7604	.8479	.1521	.9125	54° 50′
20′	.7622	.8506	.1494	.9116	40′
30′	.7640	.8533	.1467	.9107	30′
40′	.7657	.8559	.1441	.9098	20′
35° 50′	.7675	.8586	.1414	.9089	10′
36° 00′	9.7692	9.8613	10.1387	9.9080	**54° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE IV (*continued*)

→	L Sin	L Tan	L Cot	L Cos	
36° 00′	9.7692	9.8613	10.1387	9.9080	**54° 00′**
10′	.7710	.8639	.1361	.9070	53° 50′
20′	.7727	.8666	.1334	.9061	40′
30′	.7744	.8692	.1308	.9052	30′
40′	.7761	.8718	.1282	.9042	20′
36° 50′	.7778	.8745	.1255	.9033	10′
37° 00′	9.7795	9.8771	10.1229	9.9023	**53° 00′**
10′	.7811	.8797	.1203	.9014	52° 50′
20′	.7828	.8824	.1176	.9004	40′
30′	.7844	.8850	.1150	.8995	30′
40′	.7861	.8876	.1124	.8985	20′
37° 50′	.7877	.8902	.1098	.8975	10′
38° 00′	9.7893	9.8928	10.1072	9.8965	**52° 00′**
10′	.7910	.8954	.1046	.8955	51° 50′
20′	.7926	.8980	.1020	.8945	40′
30′	.7941	.9006	.0994	.8935	30′
40′	.7957	.9032	.0968	.8925	20′
38° 50′	.7973	.9058	.0942	.8915	10′
39° 00′	9.7989	9.9084	10.0916	9.8905	**51° 00′**
10′	.8004	.9110	.0890	.8895	50° 50′
20′	.8020	.9135	.0865	.8884	40′
30′	.8035	.9161	.0839	.8874	30′
40′	.8050	.9187	.0813	.8864	20′
39° 50′	.8066	.9212	.0788	.8853	10′
40° 00′	9.8081	9.9238	10.0762	9.8843	**50° 00′**
10′	.8096	.9264	.0736	.8832	49° 50′
20′	.8111	.9289	.0711	.8821	40′
30′	.8125	.9315	.0685	.8810	30′
40′	.8140	.9341	.0659	.8800	20′
40° 50′	.8155	.9366	.0634	.8789	10′
41° 00′	9.8169	9.9392	10.0608	9.8778	**49° 00′**
10′	.8184	.9417	.0583	.8767	48° 50′
20′	.8198	.9443	.0557	.8756	40′
30′	.8213	.9468	.0532	.8745	30′
40′	.8227	.9494	.0506	.8733	20′
41° 50′	.8241	.9519	.0481	.8722	10′
42° 00′	9.8255	9.9544	10.0456	9.8711	**48° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE IV (*continued*)

→	L Sin	L Tan	L Cot	L Cos	
42° 00′	9.8255	9.9544	10.0456	9.8711	**48° 00′**
10′	.8269	.9570	.0430	.8699	47° 50′
20′	.8283	.9595	.0405	.8688	40′
30′	.8297	.9621	.0379	.8676	30′
40′	.8311	.9646	.0354	.8665	20′
42° 50′	.8324	.9671	.0329	.8653	10′
43° 00′	9.8338	9.9697	10.0303	9.8641	**47° 00′**
10′	.8351	.9722	.0278	.8629	46° 50′
20′	.8365	.9747	.0253	.8618	40′
30′	.8378	.9772	.0228	.8606	30′
40′	.8391	.9798	.0202	.8594	20′
43° 50′	.8405	.9823	.0177	.8582	10′
44° 00′	9.8418	9.9848	10.0152	9.8569	**46° 00′**
10′	.8431	.9874	.0126	.8557	45° 50′
20′	.8444	.9899	.0101	.8545	40′
30′	.8457	.9924	.0076	.8532	30′
40′	.8469	.9949	.0051	.8520	20′
44° 50′	.8482	.9975	.0025	.8507	10′
45° 00′	9.8495	10.0000	10.0000	9.8495	**45° 00′**
	L Cos	L Cot	L Tan	L Sin	←

TABLE V

Squares and Square Roots

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
1.00	1.0000	1.00000	3.16228	**1.60**	2.5600	1.26491	4.00000	**2.20**	4.8400	1.48324	4.69042
1.01	1.0201	1.00499	3.17805	1.61	2.5921	1.26886	4.01248	2.21	4.8841	1.48661	4.70106
1.02	1.0404	1.00995	3.19374	1.62	2.6244	1.27279	4.02492	2.22	4.9284	1.48997	4.71169
1.03	1.0609	1.01489	3.20936	1.63	2.6569	1.27671	4.03733	2.23	4.9729	1.49332	4.72229
1.04	1.0816	1.01980	3.22490	1.64	2.6896	1.28062	4.04969	2.24	5.0176	1.49666	4.73286
1.05	1.1025	1.02470	3.24037	**1.65**	2.7225	1.28452	4.06202	**2.25**	5.0625	1.50000	4.74342
1.06	1.1236	1.02956	3.25576	1.66	2.7556	1.28841	4.07431	2.26	5.1076	1.50333	4.75395
1.07	1.1449	1.03441	3.27109	1.67	2.7889	1.29228	4.08656	2.27	5.1529	1.50665	4.76445
1.08	1.1664	1.03923	3.28634	1.68	2.8224	1.29615	4.09878	2.28	5.1984	1.50997	4.77493
1.09	1.1881	1.04403	3.30151	1.69	2.8561	1.30000	4.11096	2.29	5.2441	1.51327	4.78539
1.10	1.2100	1.04881	3.31662	**1.70**	2.8900	1.30384	4.12311	**2.30**	5.2900	1.51658	4.79583
1.11	1.2321	1.05357	3.33167	1.71	2.9241	1.30767	4.13521	2.31	5.3361	1.51987	4.80625
1.12	1.2544	1.05830	3.34664	1.72	2.9584	1.31149	4.14729	2.32	5.3824	1.52315	4.81664
1.13	1.2769	1.06301	3.36155	1.73	2.9929	1.31529	4.15933	2.33	5.4289	1.52643	4.82701
1.14	1.2996	1.06771	3.37639	1.74	3.0276	1.31909	4.17133	2.34	5.4756	1.52971	4.83735
1.15	1.3225	1.07238	3.39116	**1.75**	3.0625	1.32288	4.18330	**2.35**	5.5225	1.53297	4.84768
1.16	1.3456	1.07703	3.40588	1.76	3.0976	1.32665	4.19524	2.36	5.5696	1.53623	4.85798
1.17	1.3689	1.08167	3.42053	1.77	3.1329	1.33041	4.20714	2.37	5 6169	1.53948	4.86826
1.18	1.3924	1.08628	3.43511	1.78	3.1684	1.33417	4.21900	2.38	5.6644	1.54272	4.87852
1.19	1.4161	1.09087	3.44964	1.79	3.2041	1.33791	4.23084	2.39	5.7121	1.54596	4.88876
1.20	1.4400	1.09545	3.46410	**1.80**	3.2400	1.34164	4.24264	**2.40**	5.7600	1.54919	4.89898
1.21	1.4641	1.10000	3.47851	1.81	3.2761	1.34536	4.25441	2.41	5.8081	1.55242	4.90918
1.22	1.4884	1.10454	3.49285	1.82	3.3124	1.34907	4.26615	2.42	5.8564	1.55563	4.91935
1.23	1.5129	1.10905	3.50714	1.83	3.3489	1.35277	4.27785	2.43	5.9049	1.55885	4.92950
1.24	1.5376	1.11355	3.52136	1.84	3.3856	1.35647	4.28952	2.44	5.9536	1.56205	4.93964
1.25	1.5625	1.11803	3.53553	**1.85**	3.4225	1.36015	4.30116	**2.45**	6.0025	1.56525	4.94975
1.26	1.5876	1.12250	3.54965	1.86	3.4596	1.36382	4.31277	2.46	6.0516	1.56844	4.95984
1.27	1.6129	1.12694	3.56371	1.87	3.4969	1.36748	4.32435	2.47	6.1009	1.57162	4.96991
1.28	1.6384	1.13137	3.57771	1.88	3.5344	1.37113	4.33590	2.48	6.1504	1.57480	4.97996
1.29	1.6641	1.13578	3.59166	1.89	3.5721	1.37477	4.34741	2.49	6.2001	1.57797	4.98999
1.30	1.6900	1.14018	3.60555	**1.90**	3.6100	1.37840	4.35890	**2.50**	6.2500	1.58114	5.00000
1.31	1.7161	1.14455	3.61939	1.91	3.6481	1.38203	4.37035	2.51	6.3001	1.58430	5.00999
1.32	1.7424	1.14891	3.63318	1.92	3.6864	1.38564	4.38178	2.52	6.3504	1.58745	5.01996
1.33	1.7689	1.15326	3.64692	1.93	3 7249	1.38924	4.39318	2.53	6.4009	1.59060	5.02991
1.34	1.7956	1.15758	3.66060	1.94	3.7636	1.39284	4.40454	2.54	6.4516	1.59374	5.03984
1.35	1.8225	1.16190	3.67423	**1.95**	3.8025	1.39642	4.41588	**2.55**	6.5025	1.59687	5.04975
1.36	1.8496	1.16619	3.68782	1.96	3.8416	1.40000	4.42719	2.56	6.5536	1.60000	5.05964
1.37	1.8769	1.17047	3.70135	1.97	3.8809	1.40357	4.43847	2.57	6.6049	1.60312	5.06952
1.38	1.9044	1.17473	3.71484	1.98	3.9204	1.40712	4.44972	2.58	6.6564	1.60624	5.07937
1.39	1.9321	1.17898	3.72827	1.99	3.9601	1.41067	4.46094	2.59	6.7081	1.60935	5.08920
1.40	1.9600	1.18322	3.74166	**2.00**	4.0000	1.41421	4.47214	**2.60**	6.7600	1.61245	5.09902
1.41	1.9881	1.18743	3.75500	2.01	4.0401	1.41774	4.48330	2.61	6.8121	1.61555	5.10882
1.42	2.0164	1.19164	3.76829	2.02	4.0804	1.42127	4.49444	2.62	6.8644	1.61864	5.11859
1.43	2.0449	1.19583	3.78153	2.03	4.1209	1.42478	4.50555	2.63	6.9169	1.62173	5.12835
1.44	2.0736	1.20000	3.79473	2.04	4.1616	1.42829	4.51664	2.64	6.9696	1.62481	5.13809
1.45	2.1025	1.20416	3.80789	**2.05**	4.2025	1.43178	4.52769	**2.65**	7.0225	1.62788	5.14782
1.46	2.1316	1.20830	3.82099	2.06	4.2436	1.43527	4.53872	2.66	7.0756	1.63095	5.15752
1.47	2.1609	1.21244	3.83406	2.07	4.2849	1.43875	4.54973	2.67	7.1289	1.63401	5.16720
1.48	2.1904	1.21655	3.84708	2.08	4.3264	1.44222	4.56070	2.68	7.1824	1.63707	5.17687
1.49	2.2201	1.22066	3.86005	2.09	4.3681	1.44568	4.57165	2.69	7.2361	1.64012	5.18652
1.50	2.2500	1.22474	3.87298	**2.10**	4.4100	1.44914	4.58258	**2.70**	7.2900	1.64317	5.19615
1.51	2.2801	1.22882	3.88587	2.11	4.4521	1.45258	4.59347	2.71	7.3441	1.64621	5.20577
1.52	2.3104	1.23288	3.89872	2.12	4.4944	1.45602	4.60435	2.72	7.3984	1.64924	5.21536
1.53	2.3409	1.23693	3.91152	2.13	4.5369	1.45945	4.61519	2.73	7.4529	1.65227	5.22494
1.54	2.3716	1.24097	3.92428	2.14	4.5796	1.46287	4.62601	2.74	7.5076	1.65529	5.23450
1.55	2.4025	1.24499	3.93700	**2.15**	4.6225	1.46629	4.63681	**2.75**	7.5625	1.65831	5.24404
1.56	2.4336	1.24900	3.94968	2.16	4.6656	1.46969	4.64758	2.76	7.6176	1.66132	5.25357
1.57	2.4649	1.25300	3.96232	2.17	4.7089	1.47309	4.65833	2.77	7.6729	1.66433	5.26308
1.58	2.4964	1.25698	3.97492	2.18	4.7524	1.47648	4.66905	2.78	7.7284	1.66733	5.27257
1.59	2.5281	1.26095	3.98748	2.19	4.7961	1.47986	4.67974	2.79	7.7841	1.67033	5.28205
1.60	2.5600	1.26491	4.00000	**2.20**	4.8400	1.48324	4.69042	**2.80**	7.8400	1.67332	5.29150
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

TABLE V (*continued*)

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
2.80	7.8400	1.67332	5.29150	**3.40**	11.5600	1.84391	5.83095	**4.00**	16.0000	2.00000	6.32456
2.81	7.8961	1.67631	5.30094	3.41	11.6281	1.84662	5.83952	4.01	16.0801	2.00250	6.33246
2.82	7.9524	1.67929	5.31037	3.42	11.6964	1.84932	5.84808	4.02	16.1604	2.00499	6.34035
2.83	8.0089	1.68226	5.31977	3.43	11.7649	1.85203	5.85662	4.03	16.2409	2.00749	6.34823
2.84	8.0656	1.68523	5.32917	3.44	11.8336	1.85472	5.86515	4.04	16.3216	2.00998	6.35610
2.85	8.1225	1.68819	5.33854	**3.45**	11.9025	1.85742	5.87367	**4.05**	16.4025	2.01246	6.36396
2.86	8.1796	1.69115	5.34790	3.46	11.9716	1.86011	5.88218	4.06	16.4836	2.01494	6.37181
2.87	8.2369	1.69411	5.35724	3.47	12.0409	1.86279	5.89067	4.07	16.5649	2.01742	6.37966
2.88	8.2944	1.69706	5.36656	3.48	12.1104	1.86548	5.89915	4.08	16.6464	2.01990	6.38749
2.89	8.3521	1.70000	5.37587	3.49	12.1801	1.86815	5.90762	4.09	16.7281	2.02237	6.39531
2.90	8.4100	1 70294	5.38516	**3.50**	12.2500	1.87083	5.91608	**4.10**	16.8100	2.02485	6.40312
2.91	8.4681	1.70587	5.39444	3.51	12.3201	1.87350	5.92453	4.11	16.8921	2.02731	6.41093
2.92	8.5264	1.70880	5.40370	3.52	12.3904	1.87617	5.93296	4.12	16.9744	2.02978	6.41872
2.93	8.5849	1.71172	5.41295	3.53	12.4609	1.87883	5.94138	4.13	17.0569	2.03224	6.42651
2.94	8.6436	1.71464	5.42218	3.54	12.5316	1.88149	5.94979	4.14	17.1396	2.03470	6.43428
2.95	8.7025	1.71756	5.43139	**3.55**	12.6025	1.88414	5.95819	**4.15**	17.2225	2.03715	6.44205
2.96	8.7616	1.72047	5.44059	3.56	12.6736	1.88680	5.96657	4.16	17.3056	2.03961	6.44981
2.97	8.8209	1.72337	5.44977	3.57	12.7449	1.88944	5.97495	4.17	17.3889	2.04206	6.45755
2.98	8.8804	1.72627	5.45894	3.58	12.8164	1.89209	5.98331	4.18	17.4724	2.04450	6.46529
2.99	8.9401	1.72916	5.46809	3.59	12.8881	1.89473	5.99166	4.19	17.5561	2.04695	6.47302
3.00	9.0000	1.73205	5.47723	**3.60**	12.9600	1.89737	6.00000	**4.20**	17.6400	2.04939	6.48074
3.01	9.0601	1.73494	5.48635	3.61	13.0321	1.90000	6.00833	4.21	17.7241	2.05183	6.48845
3.02	9.1204	1.73781	5.49545	3.62	13.1044	1.90263	6.01664	4.22	17.8084	2.05426	6.49615
3.03	9.1809	1.74069	5.50454	3.63	13.1769	1.90526	6.02495	4.23	17.8929	2.05670	6.50384
3.04	9.2416	1.74356	5.51362	3.64	13.2496	1.90788	6.03324	4.24	17.9776	2.05913	6.51153
3.05	9.3025	1.74642	5.52268	**3.65**	13.3225	1.91050	6.04152	**4.25**	18.0625	2.06155	6.51920
3.06	9.3636	1.74929	5.53173	3.66	13.3956	1.91311	6.04979	4.26	18.1476	2.06398	6.52687
3.07	9.4249	1.75214	5.54076	3.67	13.4689	1.91572	6.05805	4.27	18.2329	2.06640	6.53452
3.08	9.4864	1.75499	5.54977	3.68	13.5424	1.91833	6.06630	4.28	18.3184	2.06882	6.54217
3.09	9.5481	1.75784	5.55878	3.69	13.6161	1.92094	6.07454	4.29	18.4041	2.07123	6.54981
3.10	9.6100	1.76068	5.56776	**3.70**	13.6900	1.92354	6.08276	**4.30**	18.4900	2.07364	6.55744
3.11	9.6721	1.76352	5.57674	3.71	13.7641	1.92614	6.09098	4.31	18.5761	2.07605	6.56506
3.12	9.7344	1.76635	5.58570	3.72	13.8384	1.92873	6.09918	4.32	18.6624	2.07846	6.57267
3.13	9.7969	1.76918	5.59464	3.73	13.9129	1.93132	6.10737	4.33	18.7489	2.08087	6.58027
3.14	9.8596	1.77200	5.60357	3.74	13.9876	1.93391	6.11555	4.34	18.8356	2.08327	6.58787
3.15	9.9225	1.77482	5.61249	**3.75**	14.0625	1.93649	6.12372	**4.35**	18.9225	2.08567	6.59545
3.16	9.9856	1.77764	5.62139	3.76	14.1376	1.93907	6.13188	4.36	19.0096	2.08806	6.60303
3.17	10.0489	1.78045	5.63028	3.77	14.2129	1.94165	6.14003	4.37	19.0969	2.09045	6.61060
3.18	10.1124	1.78326	5.63915	3.78	14.2884	1.94422	6.14817	4.38	19.1844	2.09284	6.61816
8.19	10.1761	1.78606	5.64801	3.79	14.3641	1.94679	6.15630	4.39	19.2721	2.09523	6.62571
3.20	10.2400	1.78885	5.65685	**3.80**	14.4400	1.94936	6.16441	**4.40**	19.3600	2.09762	6.63325
3.21	10.3041	1.79165	5.66569	3.81	14.5161	1.95192	6.17252	4.41	19.4481	2.10000	6.64078
3.22	10.3684	1.79444	5.67450	3.82	14.5924	1.95448	6.18061	4.42	19.5364	2.10238	6.64831
3.23	10.4329	1.79722	5.68331	3.83	14.6689	1.95704	6.18870	4.43	19.6249	2.10476	6.65582
3.24	10.4976	1.80000	5.69210	3.84	14.7456	1.95959	6.19677	4.44	19.7136	2.10713	6.66333
3.25	10.5625	1.80278	5.70088	**3.85**	14.8225	1.96214	6.20484	**4.45**	19.8025	2.10950	6.67083
3.26	10.6276	1.80555	5.70964	3.86	14.8996	1.96469	6.21289	4.46	19.8916	2.11187	6.67832
3.27	10.6929	1.80831	5.71839	3.87	14.9769	1.96723	6.22093	4.47	19.9809	2.11424	6.68581
3.28	10.7584	1.81108	5.72713	3.88	15.0544	1.96977	6.22896	4.48	20.0704	2.11660	6.69328
3.29	10.8241	1.81384	5.73585	3.89	15.1321	1.97231	6.23699	4.49	20.1601	2.11896	6.70075
3.30	10.8900	1.81659	5.74456	**3.90**	15.2100	1.97484	6.24500	**4.50**	20.2500	2.12132	6.70820
3.31	10.9561	1.81934	5.75326	3.91	15.2881	1.97737	6.25300	4.51	20.3401	2.12368	6.71565
3.32	11.0224	1.82209	5.76194	3.92	15.3664	1.97990	6.26099	4.52	20.4304	2.12603	6.72309
3.33	11.0889	1.82483	5.77062	3.93	15.4449	1.98242	6.26897	4.53	20.5209	2.12838	6.73053
3.34	11.1556	1.82757	5.77927	3.94	15.5236	1.98494	6.27694	4.54	20.6116	2.13073	6.73795
3.35	11.2225	1.83030	5.78792	**3.95**	15.6025	1.98746	6.28490	**4.55**	20.7025	2.13307	6.74537
3.36	11.2896	1.83303	5.79655	3.96	15.6816	1.98997	6.29285	4.56	20.7936	2.13542	6.75278
3.37	11 3569	1.83576	5.80517	3.97	15.7609	1.99249	6.30079	4.57	20.8849	2.13776	6.76018
3.38	11.4244	1.83848	5.81378	3.98	15.8404	1.99499	6.30872	4.58	20.9764	2.14009	6.76757
3.39	11.4921	1.84120	5.82237	3.99	15.9201	1.99750	6.31664	4.59	21.0681	2.14243	6.77495
3.40	11.5600	1.84391	5.83095	**4.00**	16.0000	2.00000	6.32456	**4.60**	21.1600	2.14476	6.78233
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

TABLE V (*continued*)

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
4.60	21.1600	2.14476	6.78233	**5.20**	27.0400	2.28035	7.21110	**5.80**	33.6400	2.40832	7.61577
4.61	21.2521	2.14709	6.78970	5.21	27.1441	2.28254	7.21803	5.81	33.7561	2.41039	7.62234
4.62	21.3444	2.14942	6.79706	5.22	27.2484	2.28473	7.22496	5.82	33.8724	2.41247	7.62889
4.63	21.4369	2.15174	6.80441	5.23	27.3529	2.28692	7.23187	5.83	33.9889	2.41454	7.63544
4.64	21.5296	2.15407	6.81175	5.24	27.4576	2.28910	7.23878	5.84	34.1056	2.41661	7.64199
4.65	21.6225	2.15639	6.81909	**5.25**	27.5625	2.29129	7.24569	**5.85**	34.2225	2.41868	7.64853
4.66	21.7156	2.15870	6.82642	5.26	27.6676	2.29347	7.25259	5.86	34.3396	2.42074	7.65506
4.67	21.8089	2.16102	6.83374	5.27	27.7729	2.29565	7.25948	5.87	34.4569	2.42281	7.66159
4.68	21.9024	2.16333	6.84105	5.28	27.8784	2.29783	7.26636	5.88	34.5744	2.42487	7.66812
4.69	21.9961	2.16564	6.84836	5.29	27.9841	2.30000	7.27324	5.89	34.6921	2.42693	7.67463
4.70	22.0900	2.16795	6.85565	**5.30**	28.0900	2.30217	7.28011	**5.90**	34.8100	2.42899	7.68115
4.71	22.1841	2.17025	6.86294	5.31	28.1961	2.30434	7.28697	5.91	34.9281	2.43105	7.68765
4.72	22.2784	2.17256	6.87023	5.32	28.3024	2.30651	7.29383	5.92	35.0464	2.43311	7.69415
4.73	22.3729	2.17486	6.87750	5.33	28.4089	2.30868	7.30068	5.93	35.1649	2.43516	7.70065
4.74	22.4676	2.17715	6.88477	5.34	28.5156	2.31084	7.30753	5.94	35.2836	2.43721	7.70714
4.75	22.5625	2.17945	6.89202	**5.35**	28.6225	2.31301	7.31437	**5.95**	35.4025	2.43926	7.71362
4.76	22.6576	2.18174	6.89928	5.36	28.7296	2.31517	7.32120	5.96	35.5216	2.44131	7.72010
4.77	22.7529	2.18403	6.90652	5.37	28.8369	2.31733	7.32803	5.97	35.6409	2.44336	7.72658
4.78	22.8484	2.18632	6.91375	5.38	28.9444	2.31948	7.33485	5.98	35.7604	2.44540	7.73305
4.79	22.9441	2.18861	6.92098	5.39	29.0521	2.32164	7.34166	5.99	35.8801	2.44745	7.73951
4.80	23.0400	2.19089	6.92820	**5.40**	29.1600	2.32379	7.34847	**6.00**	36.0000	2.44949	7.74597
4.81	23.1361	2.19317	6.93542	5.41	29.2681	2.32594	7.35527	6.01	36.1201	2.45153	7.75242
4.82	23.2324	2.19545	6.94262	5.42	29.3764	2.32809	7.36206	6.02	36.2404	2.45357	7.75887
4.83	23.3289	2.19773	6.94982	5.43	29.4849	2.33024	7.36885	6.03	36.3609	2.45561	7.76531
4.84	23.4256	2.20000	6.95701	5.44	29.5936	2.33238	7.37564	6.04	36.4816	2.45764	7.77174
4.85	23.5225	2.20227	6.96419	**5.45**	29.7025	2.33452	7.38241	**6.05**	36.6025	2.45967	7.77817
4.86	23.6196	2.20454	6.97137	5.46	29.8116	2.33666	7.38918	6.06	36.7236	2.46171	7.78460
4.87	23.7169	2.20681	6.97854	5.47	29.9209	2.33880	7.39594	6.07	36.8449	2.46374	7.79102
4.88	23.8144	2.20907	6.98570	5.48	30.0304	2.34094	7.40270	6.08	36.9664	2.46577	7.79744
4.89	23.9121	2.21133	6.99285	5.49	30.1401	2.34307	7.40945	6.09	37.0881	2.46779	7.80385
4.90	24.0100	2.21359	7.00000	**5.50**	30.2500	2.34521	7.41620	**6.10**	37.2100	2.46982	7.81025
4.91	24.1081	2.21585	7.00714	5.51	30.3601	2.34734	7.42294	6.11	37.3321	2.47184	7.81665
4.92	24.2064	2.21811	7.01427	5.52	30.4704	2.34947	7.42967	6.12	37.4544	2.47386	7.82304
4.93	24.3049	2.22036	7.02140	5.53	30.5809	2.35160	7.43640	6.13	37.5769	2.47588	7.82943
4.94	24.4036	2.22261	7.02851	5.54	30.6916	2.35372	7.44312	6.14	37.6996	2.47790	7.83582
4.95	24.5025	2.22486	7.03562	**5.55**	30.8025	2.35584	7.44983	**6.15**	37.8225	2.47992	7.84219
4.96	24.6016	2.22711	7.04273	5.56	30.9136	2.35797	7.45654	6.16	37.9456	2.48193	7.84857
4.97	24.7009	2.22935	7.04982	5.57	31.0249	2.36008	7.46324	6.17	38.0689	2.48395	7.85493
4.98	24.8004	2.23159	7.05691	5.58	31.1364	2.36220	7.46994	6.18	38.1924	2.48596	7.86130
4.99	24.9001	2.23383	7.06399	5.59	31.2481	2.36432	7.47663	6.19	38.3161	2.48797	7.86766
5.00	25.0000	2.23607	7.07107	**5.60**	31.3600	2.36643	7.48331	**6.20**	38.4400	2.48998	7.87401
5.01	25.1001	2.23830	7.07814	5.61	31.4721	2.36854	7.48999	6.21	38.5641	2.49199	7.88036
5.02	25.2004	2.24054	7.08520	5.62	31.5844	2.37065	7.49667	6.22	38.6884	2.49399	7.88670
5.03	25.3009	2.24277	7.09225	5.63	31.6969	2.37276	7.50333	6.23	38.8129	2.49600	7.89303
5.04	25.4016	2.24499	7.09930	5.64	31.8096	2.37487	7.50999	6.24	38.9376	2.49800	7.89937
5.05	25.5025	2.24722	7.10634	**5.65**	31.9225	2.37697	7.51665	**6.25**	39.0625	2.50000	7.90569
5.06	25.6036	2.24944	7.11337	5.66	32.0356	2.37908	7.52330	6.26	39.1876	2.50200	7.91202
5.07	25.7049	2.25167	7.12039	5.67	32.1489	2.38118	7.52994	6.27	39.3129	2.50400	7.91833
5.08	25.8064	2.25389	7.12741	5.68	32.2624	2.38328	7.53658	6.28	39.4384	2.50599	7.92465
5.09	25.9081	2.25610	7.13442	5.69	32.3761	2.38537	7.54321	6.29	39.5641	2.50799	7.93095
5.10	26.0100	2.25832	7.14143	**5.70**	32.4900	2.38747	7.54983	**6.30**	39.6900	2.50998	7.93725
5.11	26.1121	2.26053	7.14843	5.71	32.6041	2.38956	7.55645	6.31	39.8161	2.51197	7.94355
5.12	26.2144	2.26274	7.15542	5.72	32.7184	2.39165	7.56307	6.32	39.9424	2.51396	7.94984
5.13	26.3169	2.26495	7.16240	5.73	32.8329	2.39374	7.56968	6.33	40.0689	2.51595	7.95613
5.14	26.4196	2.26716	7.16938	5.74	32.9476	2.39583	7.57628	6.34	40.1956	2.51794	7.96241
5.15	26.5225	2.26936	7.17635	**5.75**	33.0625	2.39792	7.58288	**6.35**	40.3225	2.51992	7.96869
5.16	26.6256	2.27156	7.18331	5.76	33.1776	2.40000	7.58947	6.36	40.4496	2.52190	7.97496
5.17	26.7289	2.27376	7.19027	5.77	33.2929	2.40208	7.59605	6.37	40.5769	2.52389	7.98123
5.18	26.8324	2.27596	7.19722	5.78	33.4084	2.40416	7.60263	6.38	40.7044	2.52587	7.98749
5.19	26.9361	2.27816	7.20417	5.79	33.5241	2.40624	7.60920	6.39	40.8321	2.52784	7.99375
5.20	27.0400	2.28035	7.21110	**5.80**	33.6400	2.40832	7.61577	**6.40**	40.9600	2.52982	8.00000
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

TABLE V (*continued*)

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
6.40	40.9600	2.52982	8.00000	**7.00**	49.0000	2.64575	8.36660	**7.60**	57.7600	2.75681	8.71780
6.41	41.0881	2.53180	8.00625	7.01	49.1401	2.64764	8.37257	7.61	57.9121	2.75862	8.72353
6.42	41.2164	2.53377	8.01249	7.02	49.2804	2.64953	8.37854	7.62	58.0644	2.76043	8.72926
6.43	41.3449	2.53574	8.01873	7.03	49.4209	2.65141	8.38451	7.63	58.2169	2.76225	8.73499
6.44	41.4736	2.53772	8.02496	7.04	49.5616	2.65330	8.39047	7.64	58.3696	2.76405	8.74071
6.45	41.6025	2.53969	8.03119	**7.05**	49.7025	2.65518	8.39643	**7.65**	58.5225	2.76586	8.74643
6.46	41.7316	2.54165	8.03741	7.06	49.8436	2.65707	8.40238	7.66	58.6756	2.76767	8.75214
6.47	41.8609	2.54362	8.04363	7.07	49.9849	2.65895	8.40833	7.67	58.8289	2.76948	8.75785
6.48	41.9904	2.54558	8.04984	7.08	50.1264	2.66083	8.41427	7.68	58.9824	2.77128	8.76356
6.49	42.1201	2.54755	8.05605	7.09	50.2681	2.66271	8.42021	7.69	59.1361	2.77308	8.76926
6.50	42.2500	2.54951	8.06226	**7.10**	50.4100	2.66458	8.42615	**7.70**	59.2900	2.77489	8.77496
6.51	42.3801	2.55147	8.06846	7.11	50.5521	2.66646	8.43208	7.71	59.4441	2.77669	8.78066
6.52	42.5104	2.55343	8.07465	7.12	50.6944	2.66833	8.43801	7.72	59.5984	2.77849	8.78635
6.53	42.6409	2.55539	8.08084	7.13	50.8369	2.67021	8.44393	7.73	59.7529	2.78029	8.79204
6.54	42.7716	2.55734	8.08703	7.14	50.9796	2.67208	8.44985	7.74	59.9076	2.78209	8.79773
6.55	42.9025	2.55930	8.09321	**7.15**	51.1225	2.67395	8.45577	**7.75**	60.0625	2.78388	8.80341
6.56	43.0336	2.56125	8.09938	7.16	51.2656	2.67582	8.46168	7.76	60.2176	2.78568	8.80909
6.57	43.1649	2.56320	8.10555	7.17	51.4089	2.67769	8.46759	7.77	60.3729	2.78747	8.81476
6.58	43.2964	2.56515	8.11172	7.18	51.5524	2.67955	8.47349	7.78	60.5284	2.78927	8.82043
6.59	43.4281	2.56710	8.11788	7.19	51.6961	2.68142	8.47939	7.79	60.6841	2.79106	8.82610
6.60	43.5600	2.56905	8.12404	**7.20**	51.8400	2.68328	8.48528	**7.80**	60.8400	2.79285	8.83176
6.61	43.6921	2.57099	8.13019	7.21	51.9841	2.68514	8.49117	7.81	60.9961	2.79464	8.83742
6.62	43.8244	2.57294	8.13634	7.22	52.1284	2.68701	8.49706	7.82	61.1524	2.79643	8.84308
6.63	43.9569	2.57488	8.14248	7.23	52.2729	2.68887	8.50294	7.83	61.3089	2.79821	8.84873
6.64	44.0896	2.57682	8.14862	7.24	52.4176	2.69072	8.50882	7.84	61.4656	2.80000	8.85438
6.65	44.2225	2.57876	8.15475	**7.25**	52.5625	2.69258	8.51469	**7.85**	61.6225	2.80179	8.86002
6.66	44.3556	2.58070	8.16088	7.26	52.7076	2.69444	8.52056	7.86	61.7796	2.80357	8.86566
6.67	44.4889	2.58263	8.16701	7.27	52.8529	2.69629	8.52643	7.87	61.9369	2.80535	8.87130
6.68	44.6224	2.58457	8.17313	7.28	52.9984	2.69815	8.53229	7.88	62.0944	2.80713	8.87694
6.69	44.7561	2.58650	8.17924	7.29	53.1441	2.70000	8.53815	7.89	62.2521	2.80891	8.88257
6.70	44.8900	2.58844	8.18535	**7.30**	53.2900	2.70185	8.54400	**7.90**	62.4100	2.81069	8.88819
6.71	45.0241	2.59037	8.19146	7.31	53.4361	2.70370	8.54985	7.91	62.5681	2.81247	8.89382
6.72	45.1584	2.59230	8.19756	7.32	53.5824	2.70555	8.55570	7.92	62.7264	2.81425	8.89944
6.73	45.2929	2.59422	8.20366	7.33	53.7289	2.70740	8.56154	7.93	62.8849	2.81603	8.90505
6.74	45.4276	2.59615	8.20975	7.34	53.8756	2.70924	8.56738	7.94	63.0436	2.81780	8.91067
6.75	45.5625	2.59808	8.21584	**7.35**	54.0225	2.71109	8.57321	**7.95**	63.2025	2.81957	8.91628
6.76	45.6976	2.60000	8.22192	7.36	54.1696	2.71293	8.57904	7.96	63.3616	2.82135	8.92188
6.77	45.8329	2.60192	8.22800	7.37	54.3169	2.71477	8.58487	7.97	63.5209	2.82312	8.92749
6.78	45.9684	2.60384	8.23408	7.38	54.4644	2.71662	8.59069	7.98	63.6804	2.82489	8.93308
6.79	46.1041	2.60576	8.24015	7.39	54.6121	2.71846	8.59651	7.99	63.8401	2.82666	8.93868
6.80	46.2400	2.60768	8.24621	**7.40**	54.7600	2.72029	8.60233	**8.00**	64.0000	2.82843	8.94427
6.81	46.3761	2.60960	8.25227	7.41	54.9081	2.72213	8.60814	8.01	64.1601	2.83019	8.94986
6.82	46.5124	2.61151	8.25833	7.42	55.0564	2.72397	8.61394	8.02	64.3204	2.83196	8.95545
6.83	46.6489	2.61343	8.26438	7.43	55.2049	2.72580	8.61974	8.03	64.4809	2.83373	8.96103
6.84	46.7856	2.61534	8.27043	7.44	55.3536	2.72764	8.62554	8.04	64.6416	2.83549	8.96660
6.85	46.9225	2.61725	8.27647	**7.45**	55.5025	2.72947	8.63134	**8.05**	64.8025	2.83725	8.97218
6.86	47.0596	2.61916	8.28251	7.46	55.6516	2.73130	8.63713	8.06	64.9636	2.83901	8.97775
6.87	47.1969	2.62107	8.28855	7.47	55.8009	2.73313	8.64292	8.07	65.1249	2.84077	8.98332
6.88	47.3344	2.62298	8.29458	7.48	55.9504	2.73496	8.64870	8.08	65.2864	2.84253	8.98888
6.89	47.4721	2.62488	8.30060	7.49	56.1001	2.73679	8.65448	8.09	65.4481	2.84429	8.99444
6.90	47.6100	2.62679	8.30662	**7.50**	56.2500	2.73861	8.66025	**8.10**	65.6100	2.84605	9.00000
6.91	47.7481	2.62869	8.31264	7.51	56.4001	2.74044	8.66603	8.11	65.7721	2.84781	9.00555
6.92	47.8864	2.63059	8.31865	7.52	56.5504	2.74226	8.67179	8.12	65.9344	2.84956	9.01110
6.93	48.0249	2.63249	8.32466	7.53	56.7009	2.74408	8.67756	8.13	66.0969	2.85132	9.01665
6.94	48.1636	2.63439	8.33067	7.54	56.8516	2.74591	8.68332	8.14	66.2596	2.85307	9.02219
6.95	48.3025	2.63629	8.33667	**7.55**	57.0025	2.74773	8.68907	**8.15**	66.4225	2.85482	9.02774
6.96	48.4416	2.63818	8.34266	7.56	57.1536	2.74955	8.69483	8.16	66.5856	2.85657	9.03327
6.97	48.5809	2.64008	8.34865	7.57	57.3049	2.75136	8.70057	8.17	66.7489	2.85832	9.03881
6.98	48.7204	2.64197	8.35464	7.58	57.4564	2.75318	8.70632	8.18	66.9124	2.86007	9.04434
6.99	48.8601	2.64386	8.36062	7.59	57.6081	2.75500	8.71206	8.19	67.0761	2.86182	9.04986
7.00	49 0000	2.64575	8.36660	**7.60**	57.7600	2.75681	8.71780	**8.20**	67.2400	2.86356	9.05539
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

TABLE V (*continued*)

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
8.20	67.2400	2.86356	9.05539	**8.80**	77.4400	2.96648	9.38083	**9.40**	88.3600	3.06594	9.69536
8.21	67.4041	2.86531	9.06091	8.81	77.6161	2.96816	9.38616	9.41	88.5481	3.06757	9.70052
8.22	67.5684	2.86705	9.06642	8.82	77.7924	2.96985	9.39149	9.42	88.7364	3.06920	9.70567
8.23	67.7329	2.86880	9.07193	8.83	77.9689	2.97153	9.39681	9.43	88.9249	3.07083	9.71082
8.24	67.8976	2.87054	9.07744	8.84	78.1456	2.97321	9.40213	9.44	89.1136	3.07246	9.71597
8.25	68.0625	2.87228	9.08295	**8.85**	78.3225	2.97489	9.40744	**9.45**	89.3025	3.07409	9.72111
8.26	68.2276	2.87402	9.08845	8.86	78.4996	2.97658	9.41276	9.46	89.4916	3.07571	9.72625
8.27	68.3929	2.87576	9.09395	8.87	78.6769	2.97825	9.41807	9.47	89.6809	3.07734	9.73139
8.28	68.5584	2.87750	9.09945	8.88	78.8544	2.97993	9.42338	9.48	89.8704	3.07896	9.73653
8.29	68.7241	2.87924	9.10494	8.89	79.0321	2.98161	9.42868	9.49	90.0601	3.08058	9.74166
8.30	68.8900	2.88097	9.11043	**8.90**	79.2100	2.98329	9.43398	**9.50**	90.2500	3.08221	9.74679
8.31	69.0561	2.88271	9.11592	8.91	79.3881	2.98496	9.43928	9.51	90.4401	3.08383	9.75192
8.32	69.2224	2.88444	9.12140	8.92	79.5664	2.98664	9.44458	9.52	90.6304	3.08545	9.75705
8.33	69.3889	2.88617	9.12688	8.93	79.7449	2.98831	9.44987	9.53	90.8209	3.08707	9.76217
8.34	69.5556	2.88791	9.13236	8.94	79.9236	2.98998	9.45516	9.54	91.0116	3.08869	9.76729
8.35	69.7225	2.88964	9.13783	**8.95**	80.1025	2.99166	9.46044	**9.55**	91.2025	3.09031	9.77241
8.36	69.8896	2.89137	9.14330	8.96	80.2816	2.99333	9.46573	9.56	91.3936	3.09192	9.77753
8.37	70.0569	2.89310	9.14877	8.97	80.4609	2.99500	9.47101	9.57	91.5849	3.09354	9.78264
8.38	70.2244	2.89482	9.15423	8.98	80.6404	2.99666	9.47629	9.58	91.7764	3.09516	9.78775
8.39	70.3921	2.89655	9.15969	8.99	80.8201	2.99833	9.48156	9.59	91.9681	3.09677	9.79285
8.40	70.5600	2.89828	9.16515	**9.00**	81.0000	3.00000	9.48683	**9.60**	92.1600	3.09839	9.79796
8.41	70.7281	2.90000	9.17061	9.01	81.1801	3.00167	9.49210	9.61	92.3521	3.10000	9.80306
8.42	70.8964	2.90172	9.17606	9.02	81.3604	3.00333	9.49737	9.62	92.5444	3.10161	9.80816
8.43	71.0649	2.90345	9.18150	9.03	81.5409	3.00500	9.50263	9.63	92.7369	3.10322	9.81326
8.44	71.2336	2.90517	9.18695	9.04	81.7216	3.00666	9.50789	9.64	92.9296	3.10483	9.81835
8.45	71.4025	2.90689	9.19239	**9.05**	81.9025	3.00832	9.51315	**9.65**	93.1225	3.10644	9.82344
8.46	71.5716	2.90861	9.19783	9.06	82.0836	3.00998	9.51840	9.66	93.3156	3.10805	9.82853
8.47	71.7409	2.91033	9.20326	9.07	82.2649	3.01164	9.52365	9.67	93.5089	3.10966	9.83362
8.48	71.9104	2.91204	9.20869	9.08	82.4464	3.01330	9.52890	9.68	93.7024	3.11127	9.83870
8.49	72.0801	2.91376	9.21412	9.09	82.6281	3.01496	9.53415	9.69	93.8961	3.11288	9.84378
8.50	72.2500	2.91548	9.21954	**9.10**	82.8100	3.01662	9.53939	**9.70**	94.0900	3.11448	9.84886
8.51	72.4201	2.91719	9.22497	9.11	82.9921	3.01828	9.54463	9.71	94.2841	3.11609	9.85393
8.52	72.5904	2.91890	9.23038	9.12	83.1744	3.01993	9.54987	9.72	94.4784	3.11769	9.85901
8.53	72.7609	2.92062	9.23580	9.13	83.3569	3.02159	9.55510	9.73	94.6729	3.11929	9.86408
8.54	72.9316	2.92233	9.24121	9.14	83.5396	3.02324	9.56033	9.74	94.8676	3.12090	9.86914
8.55	73.1025	2.92404	9.24662	**9.15**	83.7225	3.02490	9.56556	**9.75**	95.0625	3.12250	9.87421
8.56	73.2736	2.92575	9.25203	9.16	83.9056	3.02655	9.57079	9.76	95.2576	3.12410	9.87927
8.57	73.4449	2.92746	9.25743	9.17	84.0889	3 02820	9.57601	9.77	95.4529	3.12570	9.88433
8.58	73.6164	2.92916	9.26283	9.18	84.2724	3.02985	9.58123	9.78	95.6484	3.12730	9.88939
8.59	73.7881	2.93087	9.26823	9.19	84.4561	3.03150	9.58645	9.79	95.8441	3.12890	9.89444
8.60	73.9600	2.93258	9.27362	**9.20**	84.6400	3.03315	9.59166	**9.80**	96.0400	3.13050	9.89949
8.61	74.1321	2.93428	9.27901	9.21	84.8241	3.03480	9.59687	9.81	96.2361	3.13209	9.90454
8.62	74.3044	2.93598	9.28440	9.22	85.0084	3.03645	9.60208	9.82	96.4324	3.13369	9.90959
8.63	74.4769	2.93769	9.28978	9.23	85.1929	3.03809	9.60729	9.83	96.6289	3.13528	9.91464
8.64	74.6496	2.93939	9.29516	9.24	85.3776	3.03974	9.61249	9.84	96.8256	3.13688	9.91968
8.65	74.8225	2.94109	9.30054	**9.25**	85.5625	3.04138	9.61769	**9.85**	97.0225	3.13847	9.92472
8.66	74.9956	2.94279	9.30591	9.26	85.7476	3.04302	9.62289	9.86	97.2196	3.14006	9.92975
8.67	75.1689	2.94449	9.31128	9.27	85.9329	3.04467	9.62808	9.87	97.4169	3.14166	9.93479
8.68	75.3424	2.94618	9.31665	9.28	86 1184	3.04631	9.63328	9.88	97.6144	3.14325	9.93982
8.69	75.5161	2.94788	9.32202	9.29	86.3041	3.04795	9.63846	9.89	97.8121	3.14484	9.94485
8.70	75.6900	2.94958	9.32738	**9.30**	86.4900	3.04959	9.64365	**9.90**	98.0100	3.14643	9.94987
8.71	75.8641	2.95127	9.33274	9.31	86.6761	3.05123	9.64883	9.91	98.2081	3.14802	9.95490
8.72	76.0384	2.95296	9.33809	9.32	86.8624	3.05287	9.65401	9.92	98.4064	3.14960	9.95992
8.73	76.2129	2.95466	9.34345	9.33	87.0489	3.05450	9.65919	9.93	98.6049	3.15119	9.96494
8.74	76.3876	2.95635	9.34880	9.34	87.2356	3.05614	9.66437	9.94	98.8036	3.15278	9.96995
8.75	76.5625	2.95804	9.35414	**9.35**	87.4225	3.05778	9.66954	**9.95**	99.0025	3.15436	9.97497
8.76	76.7376	2.95973	9.35949	9.36	87.6096	3.05941	9.67471	9.96	99.2016	3.15595	9.97998
8.77	76.9129	2.96142	9.36483	9.37	87.7969	3.06105	9.67988	9.97	99.4009	3.15753	9.98499
8.78	77.0884	2.96311	9.37017	9.38	87.9844	3.06268	9.68504	9.98	99.6004	3.15911	9.98999
8.79	77.2641	2.96479	9.37550	9.39	88.1721	3.06431	9.69020	9.99	99.8001	3.16070	9.99500
8.80	77.4400	2.96648	9.38083	**9.40**	88.3600	3.06594	9.69536	**10.00**	100.000	3.16228	10.0000
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

Appendix

B Answers to Odd-Numbered Problems

EXERCISE 1–1. PAGE 11

1. **(a)** Commutative law of addition.
(c) Associative law of multiplication.
(e) Commutative law of multiplication.

2. **(a)** 2. **(c)** 1. **(e)** 1. **(g)** -5. **(i)** -2. **(k)** -7.

3. **(a)** -6. **(c)** -35. **(e)** -10. **(g)** 2. **(i)** 29.

4. **(a)** -5. **(c)** 0. **(e)** $-2/3$. **(g)** $3b - 2a$. **(i)** $-ab$. **(k)** $3 - x$.

5. **(a)** 1. **(c)** 5/17. **(e)** 1/1.02. **(g)** $-\frac{15}{2x}\cdot$ **(i)** $x + y$. **(k)** $r - 0.1$.

EXERCISE 1–2. PAGE 15

1. **(a)** $-3, -2, 0, 4, 5$. **(c)** $-4, -2, -1, -1/3, 10$. **(e)** $-2, -3/2, 1, \sqrt{3}, 3$.

2. **(a)** $3 > 1/3$. **(c)** $\sqrt{2} > 1.414$. **(e)** $22/7 > \pi$.

3. **(a)** False. **(c)** True. **(e)** False.

4. **(a)** 0. **(c)** 0. **(e)** 2. **(g)** -6. **(i)** 0.

5. **(a)** $-1 < x < 1$. **(c)** $-a \leqq x \leqq a$. **(e)** $-2 < x < 4$.

6. **(a)** $4 < 5 < 6$. **(c)** $-1 < 0 < 1$. **(e)** $1 < \sqrt{3} < 2$.

7. $a + 7 \leqq 10$.

EXERCISE 1–3. PAGE 20

1. 32. **3.** -1. **5.** 1,000,000. **7.** -216. **9.** 2401. **11.** $\frac{1}{16}\cdot$ **13.** $\frac{16{,}384}{823{,}543}\cdot$

15. $\frac{81}{256}\cdot$ **17.** 32. **19.** $125a$. **21.** a^7. **23.** a^4b^4. **25.** a^4b^6. **27.** a^{13}.

29. $a^{2n}b^{2m}$. **31.** a^4. **33.** $\frac{1}{a^4}\cdot$ **35.** $\frac{16}{a^4}\cdot$ **37.** $5 - b$. **39.** $4b$. **41.** $3a - 4b$.

43. $-2a - 3b$. **45.** $(a - 2c)x^2 - (a + 2b)xy + (b - 1)y^2$. **47.** $4a - 5b - 2c$.

49. $a + (b + c), a - (-b - c)$. **51.** $a^2 + (c^2 - b^2), a^2 - (b^2 - c^2)$.

53. $2a + (b - 3c), 2a - (3c - b)$. **55.** $x^2 + (-y^2 - z^2), x^2 - (y^2 + z^2)$.

57. $-a^3b + (ab + b^2), -a^3b - (-ab - b^2)$.

59. $2x + (-3y - 4z), 2x - (3y + 4z)$. **61.** 12. **63.** 20. **65.** 17. **67.** 1.

69. -6. **71.** 8. **73.** 10. **75.** 13. **77.** 13. **79.** $\frac{3}{7}\cdot$ **81.** $\frac{3}{14}\cdot$

EXERCISE 1–4. PAGE 24

1. $2xy$. **3.** x^2y^2. **5.** $5a^2 - 6b^2$. **7.** $-4a$. **9.** $x^2 + 2xy$.

11. $x^3 + 3x^2 - 3x + 2$. **13.** $-4xy$. **15.** $5x^2y^2$. **17.** $3a^2 + 2b^2$.

19. $-2a + 2b - 2c$. **21.** $x^2 - 2xy + y^2$. **23.** $\frac{4}{15}x^3 - \frac{1}{2}x^2 + \frac{1}{28}xy + \frac{17}{6}y^2$.

25. $-18xy$. **27.** $-10x^3y^2$. **29.** $24a^4b^2c$. **31.** $-6x^2 - 4xy$. **33.** $12x^2y^2 + 6xy^3$.

35. $2x^2 + xy - 3y^2$. **37.** $4x^4 - 8x^3y - 4x^2y^2$. **39.** $x^3 - 2xy^2 + y^3$. **41.** $-2x$.

43. $2xy$. **45.** -1. **47.** 2. **49.** $-xy + 2x - 3y$. **51.** $x + 5$. **53.** $x + y$.

55. $x^3 + x^2y + xy^2 + y^3$. **57.** $x^2 - xy + y^2$. **59.** $x^2 + xy + y^2$. **61.** $2y - xy$.

63. $x^3 + 3x^2 + 3x + 1$; $x^2 + 2x + 1$. **65.** $2x^4 - 9x + 6$. **67.** Odd values.

EXERCISE 1–5. PAGE 30

1. $3(x+2y)$. **3.** $2(2x+7)$. **5.** $a(3x+2)$. **7.** $-x(a-2c+x)$.
9. $a(x-2y+3z)$. **11.** $y^2(5+3y-a)$. **13.** $(x-4)(x+4)$.
15. $(2x-3)(2x+3)$. **17.** $(7x-11)(7x+11)$. **19.** $(3y-a)(3y+a)$.
21. $a^2x^2(x-3a)(x+3a)$. **23.** $(0.1-b)(0.1+b)$.
25. $(7xy-12ab)(7xy+12ab)$. **27.** $x(6+x^2)(6-x^2)$.
29. $2(x+2)(x^2-2x+4)$. **31.** $(xy+z^2)(x^2y^2-xyz^2+z^4)$.
33. $(5p^2q^3+r^5)(25p^4q^6-5p^2q^3r^5+r^{10})$.
35. $(x+y^2)(x-y^2)(x^4+x^2y^4+y^8)$.
37. $3[3x^2-(2y+3z)][9x^4+3x^2(2y+3z)+(2y+3z)^2]$.
39. $(x+y-z+w)[(x+y)^2+(x+y)(z-w)+(z-w)^2]$.
41. $(6x-y)(36x^2+6xy+y^2)$. **43.** $x(6-x^2)^2$. **45.** $(xy-9)^2$.
47. $(x^2-5)(x^2-3)$. **49.** $(4x-3)(2x+1)$. **51.** $(x-2y)^2$.
53. $(x-4)(x+3)$. **55.** $(x^2+1)(x^2+2)$. **57.** $(1+15x^3)^2$.
59. $5x(x+4)(x-2)$. **61.** $(2x+1)(x-6)$. **63.** $(2x-1)(x+3)$.
65. $(6x-1)(x-6)$. **67.** $(x-0.6)^2$. **69.** $(a+3)(x+2y)$.
71. $(4x^2-5)(2x-3)$. **73.** $(x+3)(2a+y-z)$. **75.** $(x^2-3)(x+1)$.

EXERCISE 1–6. PAGE 33

1. 2. **3.** 1. **5.** $x-y$. **7.** $3x^3y$. **9.** $2xy^3z$. **11.** 1. **13.** 24. **15.** $24xyz$.
17. $36x^6y^4z^5$. **19.** x^2-4. **21.** $(x^2-49)(x-3)(x^3-8)(x^2-4)$.

EXERCISE 1–7. PAGE 35

1. **(a)** 12. **(c)** $4a^2$. **(e)** $18xy^3$. **(g)** $x-1$. **(i)** $(a-x)(b+x)$.
2. **(a)** $\frac{101}{347}$. **(c)** $\frac{3x^2y}{10}$. **(e)** $\frac{6xy^3}{3y+5}$. **(g)** $\frac{1}{4a}$. **(i)** $\frac{3x-7}{x+2}$. **(k)** $\frac{3x+2}{2x+8}$.
(m) $\frac{x^2-(y-3)^2}{4-(x-y)^2}$. **(o)** $x-3y$. **(q)** 1. **(s)** $\frac{3a+b}{2a-b}$. **(u)** $\frac{x}{1+x-2y}$.

EXERCISE 1–8. PAGE 38

1. $\frac{17}{24}$. **3.** $-\frac{94}{21}$. **5.** $-\frac{57}{20}$. **7.** $\frac{-2}{x^2-1}$. **9.** $\frac{x^3-3x^2-6}{x^3-1}$. **11.** $\frac{x^2+3x-5}{x^3}$.
13. $\frac{x^2+x+4}{(x+1)(x-1)^2}$. **15.** $\frac{3x^2+12x+5}{(x+2)(x+3)(x+4)}$. **17.** $\frac{1+x^2}{1-x}$.
19. $\frac{-x(x+5y)}{(x+y)(x-y)^2}$. **21.** $\frac{2(a^2-ab+b^2)}{a^2-b^2}$. **23.** $\frac{2x+y+3}{xy}$. **25.** $\frac{x^2-xy+y^2}{x^2-y^2}$.

EXERCISE 1–9. PAGE 41

1. $\frac{54}{775}$. **3.** $\frac{x}{14y}$. **5.** $\frac{5}{2}$. **7.** $\frac{a}{40b^3}$. **9.** $\frac{7a^3b^4}{27}$. **11.** $\frac{3xy(x-1)}{2(x+5)}$.
13. $\frac{(x-1)(x+2)}{(x-7)(x-2)}$. **15.** $\frac{(9x^2+6x+4)(2x-5)}{9x^2-4}$. **17.** $\frac{(4x+1)(x-6)}{(5x+1)(x-5)}$.

19. $\dfrac{-3(4x+5)(2x+3)}{4y^2z^2(4x^2+5)}$. 21. $\dfrac{28}{5}$. 23. $\dfrac{70}{59}$. 25. $\dfrac{3y^2}{5x}$.

27. $y^2(4x^2+2xy+y^2)$. 29. $\dfrac{3(a^2-7a+14)}{a^2+a-7}$. 31. $\dfrac{1-4x^2+4x^4}{x^2-4x^4}$.

33. $\dfrac{x+3}{x+5}$. 35. $\dfrac{1-3x^2}{5x^2-1}$. 37. $\dfrac{x^2+y^2}{xy}$. 39. 2. 41. x^2+1.

43. $\dfrac{y+2x}{y(y+x)}$. 45. $(x+y)^2$.

EXERCISE 1-10. PAGE 47

1. $x = -5/2$. 3. $x = -3/7$. 5. $x = -2$. 7. $\dfrac{-2}{15}$. 9. -11. 11. 7. 13. 10.

15. -2. 17. $y = \dfrac{4-23x}{2}$. 19. $y = \dfrac{x-1}{2}$. 21. $y = \dfrac{6x-3}{2}$.

23. $y = \dfrac{-2x+17}{4}$. 25. -5. 27. 7. 29. 1/4. 31. 3. 33. $\dfrac{-71}{2}$. 35. $\dfrac{-7}{4}$.

37. $\dfrac{237}{46}$. 39. $\dfrac{4}{11}$. 41. 4. 43. $\dfrac{3}{28}$. 45. 2. 47. $\dfrac{8}{41}$. 49. 40, 58. 51. 6, 7.

53. 921,600 sq ft. 55. 170 adults, 330 children. 57. 20°, 40°, 120°.

EXERCISE 2-1. PAGE 52

2. (a) 5. (c) 4. (e) $\sqrt{34}$. (g) $\sqrt{29}$.

3. (a) 5. (c) $\sqrt{2}$. (e) 7. (g) 4. (i) $\sqrt{a^2+b^2}$. (k) 2.

EXERCISE 2-2. PAGE 56

1. The area of a circle is a function of the radius of the circle, $A = \pi r^2$.

3. The area of a trapezoid is a function of its altitude and bases, $A = \dfrac{h}{2}(b_1 + b_2)$.

5. The volume of a cylinder is a function of its height and the radius of its base, $V = \pi r^2 h$.

7. The annual premium of a life insurance policy is a function of the applicant's age and physical condition, of the type of policy, of the company's rate policy, etc. No formula can be written.

9. -3, -5, 3, -2, $2\sqrt{2} -3$, -3, 3/4, $2y-3$, $\dfrac{2}{x}-3$, $\dfrac{1}{2x-3}$.

11. $-\dfrac{7}{32}$, 288, $6y^2+7$. 13. 0. 15. 14. 17. $p^2+q^2+r^2$. 19. 0. 21. 30.

23. 5/7. 25. 0. 27. $A = \pi r^2$, $C = 2\pi r$, $A = \dfrac{C^2}{4\pi}$, $C = 2\sqrt{\pi A}$.

29. $S = \sqrt[3]{36\pi V^2}$. 31. all x. 33. all x. 35. all x. 37. all x. 39. all x.

41. all x. 43. all x. 45. $x \neq 0, 1$. 47. $|x| \geqq 3$. 49. $x = 0$. 51. all x.

53. $x \neq 0, -2$. 55. all x. 57. $x \geqq -1$. 59. $-2 \leqq x \leqq 2$.

EXERCISE 2-3. PAGE 57

1. 3, 2.3, $|x|$. 3. 0, 0.5. 5. 1, 0, 1, 0, 1.

EXERCISE 2–4. PAGE 59

1. $y = \frac{15}{7}x$. **3.** 567. **5.** 5. **7.** 10/3. **9.** −2. **13.** 9/4, 27/8.
15. 1000/1, 100/1. **17.** 327°C. **19.** 99.5 lb, 95.2 lb. **21.** 0.0324 in.

EXERCISE 3–1. PAGE 67

1. **(a)** (1, 0). **(c)** (0, 1). **(e)** (1, 0).
2. **(a)** (0.54, 0.84). **(c)** (−0.99, 0.14). **(e)** (−0.65, −0.76).
3. **(a)** $\sqrt{2}/2$. **(c)** 1. **(e)** −1/2. **(g)** −2. **(i)** $-\sqrt{3}/2$.
7.

	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$.
(a)	—	$\pm\sqrt{3}/2$	$\pm 1/\sqrt{3}$	$\pm\sqrt{3}$	$\pm 2/\sqrt{3}$	2.
(c)	$\pm 5/13$	12/13	$\pm 5/12$	$\pm 12/5$	—	$\pm 13/5$.
(e)	$\pm\sqrt{3}/2$	$-1/2$	$\pm\sqrt{3}$	$\pm 1/\sqrt{3}$	—	$\pm 2/\sqrt{3}$.
(g)	$\pm 1/\sqrt{5}$	$\pm 2/\sqrt{5}$	1/2	—	$\pm\sqrt{5}/2$	$\pm\sqrt{5}$.
(i)	—	$\pm 4/5$	$\pm 3/4$	$\pm 4/3$	$\pm 5/4$	$-5/3$.

EXERCISE 3–3. PAGE 74

1. **(a)** 1. **(c)** 0.58. **(e)** 0.86.
2. **(a)** 0.9927. **(c)** 24.52. **(e)** −1.500. **(g)** 0.2571. **(i)** 5.798. **(k)** 1.011.

EXERCISE 3–4. PAGE 80

1. $\pi/3$. **3.** $\pi/6$. **5.** $2\pi/3$. **7.** $\pi/15$. **9.** $4\pi/3$. **11.** $2\pi/5$. **13.** $43\pi/36$.
15. $107\pi/60$. **17.** $7\pi/20$. **19.** 0.8090. **21.** 1.4358. **23.** 3.2107. **25.** 1.6323.
27. 45°. **29.** 270°. **31.** 15°. **33.** 630°. **35.** 27°. **37.** 470°23′54″. **39.** 43°43′.
57. $14\pi/3$, 1.25 radians. **59.** 14.74 in.
61. **(a)** 4 radians, **(b)** 16/9 radians, **(c)** 0.04 radian.

EXERCISE 3–5. PAGE 85

1. 0.5925. **3.** 1.092. **5.** 0.7412. **7.** −1.453. **9.** −0.2462. **11.** 0.2504.
13. 1.181. **15.** 9.010. **17.** 0.6817. **19.** 0.8437. **21.** 0.9831. **23.** 0.5154.
25. 0.2930. **27.** −9.462. **29.** −1.059. **31.** 173°10′, 353°10′.
33. 42°30′, 222°30′. **35.** 3°10′, 183°10′. **37.** 35°20′, 144°40′.
39. 56°30′, 236°30′. **41.** 264°50′, 275°10′. **43.** 83°10′, 263°10′.
45. 55°20′, 235°20′. **47.** 42°7′, 222°7′. **49.** 18°9′, 341°51′. **51.** 31°35′, 328°25′.
53. 67°4′, 247°4′. **55.** 7°30′, 187°30′. **57.** 97°36′, 262°24′. **59.** 93°11′, 273°11′.
61. 0.8016. **63.** 3.079. **65.** −100.00. **67.** 0.3459. **69.** −0.7073. **71.** 1.214.
73. 0.220, 6.060. **75.** 1.120, 4.260. **77.** 0.755, 3.895. **79.** 1.158, 5.122.
81. 1.143, 1.997. **83.** 0.574, 5.706.

EXERCISE 4–1. PAGE 96

1. $3y$. **3.** $\frac{64}{27}$. **5.** 100. **7.** 81. **9.** $\frac{1}{x^{3/4}}$. **11.** ay^2. **13.** 1.

15. $x + 2(xy)^{1/2} + y$. **17.** $\dfrac{1}{x+y}$. **19.** $\dfrac{x^{1/2}-1}{x^{1/2}+1}$. **21.** $4(5^{1/2})$. **23.** $(21)^{1/3}$.

25. $\dfrac{(xy)^{2/3}}{y}$. **27.** $\dfrac{(8x)^{1/4}}{x}$. **29.** $\dfrac{y^{1/3}}{y}$. **31.** $\dfrac{y}{9}$. **33.** $(5292)^{1/6}$. **35.** $x^{7/6}y^{5/6}$.

37. $\dfrac{9+3\sqrt{2}}{7}$. **39.** $\dfrac{-(1+\sqrt{5})}{2}$. **41.** $\dfrac{x+\sqrt{x^2-9}}{9}$.

43. $\left(\dfrac{x+\sqrt{x^2-y^2}}{y}\right)^2$. **45.** $\dfrac{(3-2x)\sqrt{2x-x^2}}{2-x}$. **47.** $\dfrac{1+x^2-\sqrt{1+x^2}}{x^2(1+x^2)}$.

49. $2(2-x^2)^{3/2}$.

EXERCISE 4–2. PAGE 100

1. 54. **3.** 3/2. **5.** 6,652,800. **7.** 5/24. **9.** $n(n-1)$. **11.** $(n+1)n$.

13. $\dfrac{n+1}{n}$. **15.** $\dfrac{n}{n+2}$. **19.** $x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1$.

21. $8a^6 - 36a^4b^2 + 54a^2b^4 - 27b^6$.

23. $x^{5/2} + 10x^{3/2} + 40x^{1/2} + 80x^{-1/2} + 80x^{-3/2} + 32x^{-5/2}$.

25. $\dfrac{y^8}{y^4} - 4\dfrac{x^4}{y^2} + 6 - 4\dfrac{y^2}{x^4} + \dfrac{y^4}{x^8}$.

27. $x^{12} - 6x^{10}y^2 + 15x^8y^4 - 20x^6y^6 + 15x^4y^8 - 6x^2y^{10} + y^{12}$.

29. $x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$.

31. $x^8 + 4x^7 + 10x^6 + 16x^5 + 19x^4 + 16x^3 + 10x^2 + 4x + 1$. **34.** $-8x^7$.

35. $7920a^8b^4$. **37.** $-14x^2$. **39.** $2^{11}\cdot 3^5\cdot 5\cdot 7\cdot 13x^{33}y^8$. **41.** $924x^3y^3$.

EXERCISE 5–1. PAGE 104

1. $\log_2 8 = 3$. **3.** $\log_3 81 = 4$. **5.** $\log_{10} 1000 = 3$.

7. $\log_{256} 2 = \dfrac{1}{8}$. **9.** $\log_{100} 10 = 0.5$. **11.** $y = \log_{10} x$. **13.** $8^2 = 64$.

15. $2^{-6} = \dfrac{1}{64}$. **17.** $7^3 = 343$. **19.** $10^4 = 10{,}000$. **21.** $4^{3/2} = 8$. **23.** 6. **25.** 2.

27. 3. **29.** 1/10. **31.** 5/2. **33.** No solution. **35.** $\log_b \dfrac{4\pi r^3}{3}$. **37.** $\log_b \dfrac{23^7}{2^{12}}$.

39. $\log_b (u - \sqrt{u^2 - a^2})$.

41. **(a)** $\log_b \pi + 1/2 \log_b l - 1/2 \log_b g$. **(b)** $2 \log_b T + \log_b g - 2 \log_b \pi$.

EXERCISE 5–2. PAGE 107

1. 1. **3.** 4. **5.** −1. **7.** −3. **9.** −1. **11.** 5. **13.** −1. **15.** −1. **17.** 7314.
19. 7.314. **21.** 7314000. **23.** 0.007314.

EXERCISE 5–3. PAGE 110

1. 1.5441. **3.** 2.0212. **5.** 7.7931-10. **7.** 4.3636. **9.** 9.5490-10. **11.** 8.8215-10.
13. 9.9279-10. **15.** 0.4536. **17.** 7.8452-10. **19.** 9.8908-10. **21.** 0.4972.
23. 8.9439-10. **25.** 9.9824-10. **27.** 9.9476-10. **29.** 9.1306-10. **31.** 46.4.
33. 0.262. **35.** 504. **37.** 0.0000000000276. **39.** 69.2. **41.** 292.3.
43. 5,454,000,000. **45.** 0.06114. **47.** 4.554. **49.** 0.00001072. **51.** 0.6021.
53. 0.4266. **55.** 1.585. **57.** 3.728. **59.** 2.5023. **61.** 1.6297.

EXERCISE 5–4. PAGE 112

1. 8.540. **3.** 0.04292. **5.** 3.183. **7.** 0.0008416. **9.** 0.1104. **11.** 54.61. **13.** 48.91. **15.** 11,670. **17.** 0.1795. **19.** 0.02950. **21.** 20.56. **23.** 538,100. **25.** 1.708. **27.** -1.021. **29.** 1.249. **31.** 0.4343, 0.2171, 9.5657-10, 23.1, 22.46. **33.** 127,900,000 sq ft. **35.** 12.62 ft. **37.** 6,070,000 sq ft. **39.** 16.5 amp. **41.** $1,074.00.

EXERCISE 5–5. PAGE 114

1. 2.3026. **3.** 1.4429. **5.** 0.8735. **7.** 6.0001. **9.** 1.5373. **11.** 2.0794. **13.** 1.4307. **15.** 0.8228.

EXERCISE 6–1. PAGE 119

1. $B = 57°$, $b = 18$, $c = 22$. **3.** $A = 18°50'$, $a = 21$, $c = 66$. **5.** $A = 27°1'$, $B = 62°59'$, $c = 7.012$. **7.** $A = 22°44'$, $b = 10.30$, $c = 11.17$. **9.** $B = 53°39'$, $b = 13.40$, $c = 16.64$. **11.** $B = 46°43'$, $a = 73.66$, $c = 107.5$. **13.** $A = 8°58'$, $B = 81°2'$, $c = 793.0$. **15.** $A = 49°3'$, $a = 2.663$, $c = 3.528$. **17.** $h = 29$ ft, $l = 33$ ft. **19.** 113 ft. **21.** 227 ft. **23.** 13°34′. **25.** 4 ft.

EXERCISE 6–2. PAGE 124

1. 63°26′, 11,000 ft. **3.** 60°. **5.** N11°24′W, 74 nautical miles. **7.** 400 ft. **9.** 80 ft, 173 ft.

EXERCISE 6–3. PAGE 131

1. **(a)** 5. **(c)** $2\sqrt{5}$. **(e)** $\sqrt{2}$. **(g)** 4.

2. **(a)** 5 [3/5, 4/5]. **(c)** $2\sqrt{5}\left[-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right]$. **(e)** $\sqrt{2}\left[\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right]$. **(g)** 4 [0, 1].

3. **(a)** 5, 53°8′. **(c)** 5, 143°8′.

4. **(a)** $\sqrt{5}$, 63°26′. **(c)** $\sqrt{73}$, 159°27′.

5. 12, 0°. **7.** 22 knots, 20 knots. **9.** 60 lb. **11.** 49 lb, 250°45′.

EXERCISE 6–4. PAGE 133

1. 9.8733-10. **3.** 8.8059-10. **5.** 0.7391. **7.** 9.3661-10. **9.** 9.9427-10. **11.** 9.5906-10. **13.** 0.0030. **15.** 0.1004. **17.** 1°20′, 181°20′. **19.** 8°30′, 171°30′. **21.** 18°, 198°. **23.** 54°16′, 234°16′. **25.** 64°2′, 295°58′. **27.** 30°11′, 210°11′. **29.** 53°40′, 233°40′. **31.** 27°24′, 152°36′. **33.** 1°27′, 181°27′. **35.** 65°47′, 294°13′.

EXERCISE 6–5. PAGE 134

1. $B = 59°$, $b = 60$, $c = 117$. **3.** $B = 64°40'$, $b = 133.9$, $c = 148.2$. **5.** $A = 71°15'$, $b = 2.01$, $a = 5.94$. **7.** $A = 62°51'$, $b = 18.53$, $c = 40.61$. **9.** 0°20′. **11.** 671 lb, 200°40′. **13.** 607 mph, N36°42′W. **15.** 1815 lb, 1962 lb. **17.** 49.19 in. **19.** 16,900 lb.

EXERCISE 7–1. PAGE 138

1. $\frac{1}{4}(\sqrt{6}-\sqrt{2})$. **3.** $\frac{\sqrt{2}}{2}$. **5.** $2-\sqrt{3}$. **7.** $2-\sqrt{3}$. **9.** $-\frac{33}{65}, \frac{56}{65}, -\frac{33}{56}$.
11. -3. **13.** $\frac{1}{6}(\sqrt{3}-2\sqrt{2}), \frac{1}{6}(1+2\sqrt{6})$. **15.** $4/5, -3/5$.
17. $33/56, 63/16$.

EXERCISE 7–2. PAGE 142

1. $\frac{1}{2}\sqrt{2-\sqrt{2}}$. **3.** $\frac{\sqrt{3}}{2}$. **5.** $\frac{1}{2}\sqrt{2+\sqrt{2}}$. **7.** $1+\sqrt{2}$.
9. (a) $-\frac{120}{169}$, (b) $\frac{\sqrt{26}}{26}$, (c) $-\frac{120}{119}$, (d) $-\frac{119}{120}$, (e) $\frac{5\sqrt{26}}{26}$, (f) $\frac{119}{169}$, (g) $\frac{2035}{2197}$, (h) $\frac{28560}{239}$.
11. (a) $-3/5$, (b) $-\frac{1}{10}\sqrt{50+5\sqrt{10}}$, (c) $3/4$, (d) $4/3$, (e) $\frac{1}{10}\sqrt{50-5\sqrt{10}}$, (f) $-4/5$, (g) $\frac{9\sqrt{10}}{50}$, (h) $\frac{24}{7}$.
13. $\cos 3\theta$. **15.** $\tan 6\theta$. **17.** $\sin^2 2\theta$. **19.** $\cos 2\theta$. **21.** $\tan 2\theta$. **23.** $\cos\theta$.
25. $\cot\frac{\theta}{4}$. **27.** $\tan^2\frac{3\theta}{2}$.

EXERCISE 7–3. PAGE 144

1. $\frac{1}{2}[\sin 7\theta - \sin\theta]$. **3.** $\sin 10\theta + \sin 2\theta$. **5.** $\frac{1}{2}[\cos 6\theta + \cos 2\theta]$.
7. $-\frac{1}{2}[\cos 48° - \cos 8°]$. **9.** $-\frac{1}{2}[\cos 6\theta - \cos 4\theta]$. **11.** $2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}$.
13. $2\sin\frac{9\theta}{2}\cos\frac{3\theta}{2}$. **15.** $-2\sin 50°\sin 30°$. **17.** $2\sin 32°30'\cos 7°30'$.
19. $2\sin 43°\cos 3°$.

EXERCISE 8–1. PAGE 154

1. 2π, 3, 0°. **3.** 2π, 1/2, 0°. **5.** $8\pi/3$, 1/3, 0°. **7.** $5\pi/2$, ∞, 0°.
9. 8 radians, 1, 0°. **11.** $2\pi/5$, 3, 0°. **13.** $\pi/6$, ∞, 0°. **15.** $\pi/4$, ∞, 0°.
17. 1 radian, ∞, $\frac{4}{\pi}$ radians. **19.** 2 radians, ∞, $\frac{7}{\pi}$ radians. **21.** 2π, 1, 0°.
23. π, 3, $\frac{\pi}{12}$ radian.

EXERCISE 8–2. PAGE 161

1. $x=\frac{3y+5}{2}$. **3.** $x=\frac{y-b}{m}$. **5.** $x=-12y-22$. **7.** $\pi/6$. **9.** $\pi/6$.
11. $\pi/2$. **13.** $\pi/3$. **15.** $\pi/6$. **17.** $-\pi/3$. **19.** $\frac{2\pi}{3}$. **21.** $-24°27'$.
23. 12/5. **25.** 5/13. **27.** 3/4. **29.** 0.3919. **31.** $\frac{5\sqrt{6}}{12}$. **33.** 3/4. **35.** $-3/5$.
37. $-\pi/2$. **39.** $-u$. **41.** u. **43.** u. **45.** $\frac{\sqrt{1+u^2}}{u}$. **47.** u. **49.** u. **51.** $1/u$.
53. $\frac{\sqrt{1-u^2}}{1-u^2}$. **55.** $\sqrt{1-u^2}$.

EXERCISE 9–1. PAGE 168

1. $x = 10/7, y = -6/7$. **3.** $x = -9/7, y = 15/7$. **5.** $x = -2, y = 1$.
7. $x = \frac{18}{13}, y = \frac{-1}{13}$. **9.** $x = \frac{-47}{20}, y = \frac{1}{20}$. **11.** $x = \frac{-10}{7}, y = 9/7$.
13. $x = \frac{-11}{75}, y = \frac{68}{75}$. **15.** Inconsistent. **17.** $x = \frac{-21}{5}, y = 7/5, z = \frac{-22}{5}$.
19. $x = \frac{96}{49}, y = \frac{-3}{7}, z = \frac{67}{49}$. **21.** $x = 10/7, y = 25/7, z = -1/7$.
23. $x = 16/11, y = -21/11, z = -9/2$. **25.** $x = 9/5, y = -1/5, z = 1$.
27. $x = 10/7, y = 4, z = 20/3$. **29.** $x = 0, y = 5/7$. **31.** Inconsistent.
33. Inconsistent. **35.** Consistent and dependent.
37. Consistent and independent. **39.** Inconsistent.
41. Inconsistent unless $c = 1$. Consistent and dependent if $c = 1$.

EXERCISE 9–2. PAGE 172

1. $(1/4, 0), (0, -1/3)$. **3.** $(4, 0), (0, -4)$. **5.** $(0, 0)$. **7.** $(-4/3, 0), (0, 4)$.
9. $(5/3, 0), (0, -5)$. **11.** $x = 6/5, y = 4/5$. **13.** $x = 23/7, y = 22/7$.
15. $x = 1, y = 5/8$. **17.** $x = -\frac{9}{14}, y = \frac{8}{7}$.

EXERCISE 10–1. PAGE 180

1. 5. **3.** 0. **5.** 0. **7.** 7. **9.** -11. **11.** 22. **13.** $x = 4.1, y = 0.3$.
15. $x = 25/13, y = -5/13$. **17.** $\left(\frac{17}{7}, 0\right), (0, -17)$.
19. $(-3, -1), (2, 4), (6, -3)$.

EXERCISE 10–2. PAGE 185

1. $x = 2, y = 3, z = -2$. **3.** $x = 1, y = -2, z = 2/3$.
5. $x = 2, y = 3, z = -2$. **7.** $x = -8z, y = -3z$. **9.** No nontrivial solution.

EXERCISE 10–3. PAGE 187

1. 0. **3.** 0. **5.** -110. **7.** 308. **9.** 2184.

EXERCISE 11–1. PAGE 193

1. $0 + 4i, 0 - 4i$. **3.** $0 - 3i, 0 + 3i$. **5.** $0 - 6|a|i, 0 + 6|a|i$.
7. $3\sqrt{2} + 3\sqrt{2}i, 3\sqrt{2} - 3\sqrt{2}i$. **9.** $1 + 4\sqrt{2}i, 1 - 4\sqrt{2}i$.
11. $\sqrt{15} + 8|a|\sqrt{ab}\, i, \sqrt{15} - 8|a|\sqrt{ab}\, i$. **13.** $-i$. **15.** $-i$. **17.** $-i$.
19. i. **21.** 1. **23.** 0. **25.** 0. **27.** $x = 3/2, y = 1/3$. **29.** $x = 6, y = -5$.
31. $x = 1, y = -4$. **33.** $x = 2, y = 5/2$. **35.** $x = 7/3, y = 4/3$. **37.** $7 + 3i$.
39. $\frac{3}{2} + \frac{\sqrt{3}}{2} i$. **41.** $-1 + 5i$. **43.** $25 + 0i$. **45.** $-1 + i$. **47.** $1 + 0i$.

49. $-3 + (\sqrt{2} + \sqrt{3})i$. **51.** $6 + 0i$. **53.** $13 + 11i$. **55.** $-8 + 4i$.
57. $11 - 3i$. **59.** $27 + 24i$. **61.** $28 + 16i$. **63.** $5 - 2\sqrt{2} + (2 + 5\sqrt{2})i$.
65. $-2 - 26i$. **67.** $\frac{1}{2} + \frac{7}{2}i$. **69.** $\frac{22}{41} - \frac{7}{41}i$. **71.** $33 - 22i$. **73.** $\frac{6}{61} + \frac{5}{61}i$.
75. $-\frac{3}{25} + \frac{4}{25}i$. **77.** $\frac{625}{3233} + \frac{1019}{3233}i$.

EXERCISE 11–2. PAGE 197

13. $3 - 2i$. **15.** $-1 - 9i$. **17.** $-2 + 3i$. **19.** $6 - 4i$. **21.** $0 + 3i$.
23. $5 + (2 - \sqrt{3})i$. **25.** $\sqrt{2}(\cos 45° + i \sin 45°)$. **27.** $3(\cos 270° + i \sin 270°)$.
29. $\frac{\sqrt{3}}{2}(\cos 54°44' + i \sin 54°44')$. **31.** $13(\cos 67°23' + i \sin 67°23')$.
33. $3(\cos 0° + i \sin 0°)$. **35.** $\frac{\sqrt{61}}{61}(\cos 320°12' + i \sin 320°12')$.

EXERCISE 11–3. PAGE 202

1. $(1 + \sqrt{3}) + (1 - \sqrt{3})i$. **3.** $-2\sqrt{3} + 2i$. **5.** $\frac{(1 - \sqrt{3})}{2} + \frac{(1 + \sqrt{3})}{2}i$.
7. $1 + i$. **9.** -1. **11.** $-1/2 - \frac{\sqrt{3}}{2}i$. **13.** $\frac{4}{25} + \frac{3}{25}i$. **15.** $1 + \frac{3}{2}i$.
17. $-2^{10}i$. **19.** $\frac{3^{50}}{2^{100}} \cdot (0.6065 + 0.7951\,i)$. **21.** -1.
23. $2[\cos (9° + k\,90°) + i \sin (9° + k\,90°)]$, $k = 0, 1, 2, 3$.
25. $2[\cos (36° + k\,72°) + i\,(36° + k\,72°)]$, $k = 0, 1, 2, 3, 4$. **27.** $1, i, -1, -i$.
29. $\cos(80° + k\,120°) + i \sin (80° + k\,120°)$, $k = 0, 1, 2$.
31. $\frac{14 + 18i}{5}$ amperes. **33.** $\frac{828 + 154i}{173}$ ohms.

EXERCISE 12–1. PAGE 206

1. $0, -7$. **3.** $-1/2, -3/4$. **5.** $3, -2$. **7.** $3, 3, -3$.
9. $\sin\theta = 0, 1$; $0°, 90°, 180°$. **11.** $\sin\theta = 1, -2$; $90°$.
13. $\sec\theta = 2, -8$; $60°, 300°, 97°11', 262°49'$.
15. $\cot\theta = -3, 2$; $161°34', 341°34', 26°34', 206°34'$.
17. $\cot\theta = 7, -17$, $8°8', 188°8', 176°38', 356°38'$.
19. $\sin\theta = 0, 2, -\frac{5}{4}$; $0°, 180°$.

EXERCISE 12–2. PAGE 209

1. $10, -2$. **3.** $10, -3$. **5.** $\frac{-1 \pm \sqrt{7}i}{2}$. **7.** $\frac{-1 \pm \sqrt{21}}{2}$.
9. $\tan\theta = 1 \pm \sqrt{2}$; $\theta = 67°30', 247°30', 157°30', 337°30'$.
11. $\sec\theta = 1, -3$; $0°, 109°28', 250°32'$.

13. $\csc\theta = \dfrac{1 \pm \sqrt{33}}{4}$; 36°23′, 143°37′, 237°30′, 302°30′.

15. $\cos\theta = 0.4142,\ -2.414$, 65°32′, 294°28′. **17.** $(x-3)^2 + 4(y+2)^2 = 4$.

19. $(x-5)^2 + 4(y-5)^2 = 16$. **21.** $4(x-2)^2 + 9(y-1)^2 = 36$.

23. $4(x+4)^2 - 9(y-2)^2 = -36$. **25.** $3(x+10/3)^2 - (y+1)^2 = 64/3$.

27. $\sqrt{2[(x-4)^2 + 9/2]}$. **29.** $\dfrac{1}{\sqrt{(x-3)^2 - 16}}$. **31.** $[2((x+7)^2 - 32)]^{-1/2}$.

33. $[9((x+4/3)^2 + 1)]^{-1/3}$.

EXERCISE 12–3. PAGE 212

1. 1, −7. **3.** $-1 \pm \sqrt{6}$. **5.** −1, −1. **7.** $\dfrac{-3 \pm \sqrt{7}i}{4}$. **9.** 5/2, −1/2.

11. −1, − 11/8. **13.** −1, −15/7.

15. $\tan\theta = 1,\ -5/2$, 45°, 225°, 111°48′, 291°48′.

17. $\sin\theta = 3.7913,\ -0.7913$; 232°18′, 307°42′.

19. $\cos\theta = \dfrac{-2 \pm \sqrt{5}i}{3}$; no values of θ.

21. $\sec\theta = 0.414,\ -2.414$; 114°28′, 245°32′.

23. $\csc\theta = -0.6972,\ -4.3028$; 193°26′, 346°34′.

EXERCISE 12–4. PAGE 213

1. 18. **3.** −4. **5.** $8 \pm 4\sqrt{5}$. **7.** 5. **9.** $\dfrac{13 - 2\sqrt{13}}{9}$. **11.** 1, 5.

EXERCISE 12–5. PAGE 215

1. $\pm\sqrt{3},\ \pm 2i$. **3.** $\pm i,\ \pm i\sqrt{6}$. **5.** ±2, ±3. **7.** $\dfrac{1 \pm 2i}{3}$.

9. $-4 \pm \sqrt{15},\ 3 \pm 2\sqrt{2}$.

EXERCISE 12–6. PAGE 216

1. Conjugate imaginary. **3.** Real, unequal, irrational.

5. Real, unequal, irrational. **7.** Real, unequal, rational.

9. Real, equal, rational. **11.** Conjugate imaginary.

13. Real, unequal, irrational. **15.** Conjugate imaginary.

EXERCISE 12–7. PAGE 218

1. −2, −1. **3.** 0, 2. **5.** 3/2, 6/5. **7.** 6/5, −1/5.

9. $\dfrac{2}{5}, \dfrac{17}{100}$. **11.** $x^2 - 2x = 0$. **13.** $x^2 - 9x + 18 = 0$. **15.** $x^2 - 4x + 9 = 0$.

17. $x^2 + 1 = 0$. **19.** $x^2 - (\sqrt{3} - \sqrt{5})x = 0$. **21.** $x^2 - 2\sqrt{3}\,x + 8 = 0$.

EXERCISE 12-8. PAGE 224

1. Circle. **3.** Intersecting lines. **5.** Hyperbola. **7.** Parabola. **9.** Intersecting lines. **11.** Hyperbola. **13.** Intersecting lines. **15.** Intersecting lines. **17.** Intersecting lines. **19.** Parallel lines.

EXERCISE 12-9. PAGE 227

1. $(3, 3), (-3/2, 3/4)$. **3.** $(3, 4), (-4, -3)$. **5.** $(2, 4), (-3, 9)$. **7.** $(3, 2), (-1, -6)$. **9.** $(2, 3)$. **11.** $(\pm 2\sqrt{3}, 1)$. **13.** No solution. **15.** $(\pm 4, \pm 2)$.

EXERCISE 12-10. PAGE 231

1. $(3, 3), (-3/2, 3/4)$. **3.** $(3, 4), (-4, -3)$. **5.** $(4, -3), (-2, 6)$.

7. $(4, 0), (-5, 3)$. **9.** $(4, 6), (-3, -1)$. **11.** $(\pm 3, 1)$ $(\pm 3\sqrt{10}i, -10)$.

13. $\left(\pm \frac{12\sqrt{10}}{5} i, \pm \frac{3\sqrt{385}}{5}\right)$. **15.** $(\pm 2, \pm 4)$. **17.** $(1, 2), (2, 1)$.

19. $(4, 1), (-4, -1), (14, -4), (-14, 4)$.

21. $(4, 2), (-4, -2), (\sqrt{6}, -2\sqrt{6},), (-\sqrt{6}, 2\sqrt{6})$. **23.** $(5, 5), (-5, -5)$.

25. $(3, 4)$ $(4, 3), (-3, -4), (-4, -3)$.

27. $(2 + i\sqrt{2}, 2 - i\sqrt{2}), (-3 + i\sqrt{7}, -3 - i\sqrt{7}), (2 - i\sqrt{2}, 2 + i\sqrt{2}), (-3 - i\sqrt{7}, -3 + i\sqrt{7})$.

29. $\left(\frac{31 - 3\sqrt{93}}{31}, \frac{31 + 3\sqrt{93}}{31}\right), \left(\frac{31 + 3\sqrt{93}}{31}, \frac{31 - 3\sqrt{93}}{31}\right)$.

EXERCISE 12-11. PAGE 233

1. 6. **3.** 4/3. **5.** 2.292. **7.** 1.682. **9.** 1. **11.** 1.836. **13.** 2.718. **15.** 8.547. **17.** 2.944. **19.** 49.3. **21.** 10, 0.1. **23.** 0.7874. **25.** −0.44.

27. $\frac{\log(1 - ac) - \log b - \log c}{\log c}$. **29.** $\log_e (y \pm \sqrt{y^2 - 1})$. **31.** 0.6544.

EXERCISE 13-1. PAGE 239

1. $Q: x - 7, R: 0$. **3.** $Q: x - 1, R: 2$. **5.** $Q: x^3 - 3x^2 + 6x - 24, R: 78$.

7. $Q: 2x^3 + 3x^2 + 4, R: 0$. **9.** $Q: x^2 + 5x + 8, R: 11$.

11. $Q: x^2 + 2x - 15, R: 0$. **13.** $Q: x^3 + 3x - 6, R: 0$.

15. $Q: x^{n-1} + x^{n-2}y + \cdots + y^{n-1}, R: 0$. **17.** 52, 2. **19.** 24, −36.

EXERCISE 13-2. PAGE 241

1. $2x^2 - 2x - 3$. **3.** Not a factor. **5.** $x^2 + 2x - 5$.

7. $x^7 + 2x^6y^2 + 4x^5y^4 + 8x^4y^6 + 16x^3y^8 + 32x^2y^{10} + 64xy^{12} + 128y^{14}$.

9. Not a factor. **11.** $5x^2 + 2ab(5 - 3ab^2)x - 12a^3b^4$. **13.** Not a factor.

15. $12x^3 - 22x^2 - 34x + 60$. **17.** $24x^3 - 90x^2 + 39x + 45$.

EXERCISE 13-3. PAGE 244

1. $1 \pm \sqrt{3}i, -3$. **3.** $1 \pm i, \pm 3$. **5.** $2 \pm \sqrt{2}\,i, 1, 1, 2$.
7. $x^3 - 4x^2 + 9x - 10 = 0$.

EXERCISE 13-4. PAGE 247

1. $-1/3, 1/2, 5/3$. **9.** -1. **11.** $2, 2, -2, -2$. **13.** $2/3, \dfrac{-1 \pm \sqrt{3}i}{2}$.
15. $1/2, \pm i$. **17.** $3/5, 1 \pm i$. **19.** $1/2, \pm \sqrt{2}$.

EXERCISE 14-1. PAGE 253

1. $x < 3$. **3.** $x > -5$. **5.** $x < 4$. **7.** $x < -1/2$. **9.** $x < -1$.
11. $-\dfrac{1}{2} < x < \dfrac{1}{2}$. **13.** $-\dfrac{5}{3} < x < \dfrac{1}{3}$. **15.** $-\dfrac{1}{2} < x < \dfrac{7}{2}$. **17.** $-13 < x < 13$.
19. $-4 \leqq x \leqq 4$. **21.** $-1 < x < 1/3$. **23.** $x < -5/3, x > 2$.
25. No values of x. **27.** $x < -3, -2 < x < -1$. **29.** $1 < x < 2, x > 3$.
31. $|x| \geqq 5$. **33.** $x < -2, x > 0$. **35.** $x < -1/2$.

EXERCISE 15-1. PAGE 260

1. 1, 2, 4, 7, 11. **3.** 1, 2, 3, 5, 11, 35. **7.** 2, 5, 11, 23, 47; 2, 7, 18, 41, 88.

EXERCISE 15-2. PAGE 263

1. 17, 20. **3.** 18, 25. **5.** Not an arithmetic progression. **7.** $4b - 3a, 5b - 4a$.
9. $\dfrac{4a - 2b}{2}, \dfrac{5a - 3b}{2}$. **11.** $l_{26} = 78, S_{26} = 1053$. **13.** $l_{10} = 10, S_{10} = 55$.
15. $l_{75} = 149, S_{75} = 5625$. **17.** $l_{20} = 5.9, S_{20} = 61$. **19.** $l_{100} = 100, S_{100} = 5050$.
21. $l_{10} = 100, S_{10} = 550$. **23.** $a = -\dfrac{48}{5}, l_{45} = \dfrac{62}{5}$.
25. $a = 1, n = 13$ or $a = -1/2, n = 16$. **27.** $l_9 = 9, n = 9$. **29.** n^2.
31. $d = \dfrac{-60}{11}$. **33.** 33, 16. **35.** $d = \dfrac{a^2 - 1}{a(k + 1)}$. **37.** \$37.75. **39.** 282.

EXERCISE 15-3. PAGE 264

1. 1/15, 1/19. **3.** 1/20, 1/25. **5.** 4, 16/5. **7.** 1/47. **9.** $\dfrac{10}{9}, \dfrac{5}{4}, \dfrac{10}{7}, \dfrac{5}{3}$.
11. $\dfrac{120}{17}, \dfrac{60}{7}, \dfrac{120}{11}, 15$. **13.** 36.

EXERCISE 15-4. PAGE 267

1. 128, 512. **3.** 256, 1024. **5.** 8, 16/3. **7.** $\dfrac{\sqrt{6}}{12}, \dfrac{\sqrt{3}}{12}$. **9.** $-1, 1$.
11. $l_{15} = \dfrac{2^{14}}{3^{13}}, S_{15} = 9[1 - (2/3)^{15}]$. **13.** $l_{101} = 10^{-98}, S_{101} = \dfrac{1000}{11}[1 + 10^{-101}]$.
15. $l_6 = \dfrac{\log 3}{16}, S_6 = \dfrac{63 \log 3}{16}$. **17.** $3, -14$. **19.** $\dfrac{3 \pm \sqrt{17}}{2}$. **21.** $\pm 6, 15/2$.

23. 10; 100; 1000; 10,000; 100,000.

27. As n increases, the sum approaches 3 as a limit. **29.** $\dfrac{1 - x^n}{(1 - x)^2} - \dfrac{nx^n}{1 - x}$.

EXERCISE 15–5. PAGE 270

1. 64/65. **3.** $\dfrac{9 + 3\sqrt{3}}{2}$. **5.** 16. **7.** 3. **9.** 5/8. **11.** 2/11.

13. 36, $36[1 - (1/3)^{20}]$, $36(1/3)^{20}$. **15.** 12 ft, $12[1 - (3/4)^{10}]$ feet. **17.** 1/9.

19. 10/11. **21.** 1/6. **23.** 3. **25.** $\dfrac{3809}{33000}$.

EXERCISE 15–6. PAGE 272

1. $1 - x + x^2 - x^3$. **3.** $1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16}$. **5.** $1 - \dfrac{x}{2} + \dfrac{3x^2}{8} - \dfrac{5x^3}{16}$.

7. $\dfrac{1}{x} - \dfrac{y}{x^2} + \dfrac{y^2}{x^3} - \dfrac{y^3}{x^4}$. **9.** $\dfrac{1}{x^3} - \dfrac{3y}{x^4} + \dfrac{6y^2}{x^5} - \dfrac{10y^3}{x^6}$. **11.** 1.2190. **13.** 1.3684.

15. 0.8508. **17.** 0.9415. **19.** 1.0149.

EXERCISE 17–1. PAGE 279

1. 72. **3.** 216. **5.** 9999. **7.** 20,160; 7560.

EXERCISE 17–2. PAGE 282

1. 20; 210; 95,040; 143,640; 970,200. **3.** (a) 720, (b) 48, (c) 480. **5.** 125.

7. ${}_8C_0a^8 + {}_8C_1a^7b + {}_8C_2a^6b^2 + {}_8C_3a^5b^3 + {}_8C_4a^4b^4 + {}_8C_5a^3b^5 + {}_8C_6a^2b^6 + {}_8C_7ab^7 + {}_8C_8b^8$.

9. 9,979,200. **11.** 66. **13.** 63. **15.** $\dfrac{52!}{(13!)^4}$.

EXERCISE 17–3. PAGE 288

1. 2/5, \$24.00. **3.** 1/3, 2/3. **5.** \$7.29. **7.** \$2.42. **9.** 0.553. **11.** 16/63.

13. $\dfrac{1}{1728}$. **15.** $\dfrac{5}{72}, \dfrac{215}{216}$.

EXERCISE 18–1. PAGE 295

1. $C = 98°58'$, $b = 14.55$, $c = 20.46$. **3.** $A = 89°27'$, $a = 1169$, $b = 1079$.

5. $B = 5°31'$, $b = 1.051$, $c = 7.513$. **7.** $A = 17°3'$, $B = 100°26'$, $b = 71.18$.

9. $B = 24°27'$, $C = 101°20'$, $c = 1193$. **11.** No solution. **13.** 615.3 ft.

15. 449 ft. **17.** 12.6 in. **19.** 3158 lb, 101°37′.

EXERCISE 18–2. PAGE 298

1. $c = 270$, $B = 51°30'$, $A = 69°50'$. **3.** $a = 290$, $B = 11°50'$, $C = 101°$.

5. $a = 160$, $B = 10°$, $C = 20°$. **7.** 48°20′. **9.** No solution.

11. $A = 54°20'$, $B = 59°40'$, $C = 66°$. **13.** 3257 ft. **15.** 700 ft.

17. $c = 20$, $A = 63°$, $B = 73°$, $C = 44°$.

EXERCISE 18-3. PAGE 301

1. $A = 50°$, $B = 70°$, $c = 56$. **3.** $A = 61°$, $C = 23°50'$, $b = 262$.
5. $B = 31°11'$, $C = 70°$, $a = 79.25$. **7.** $A = 47°$, $B = 58°$, $C = 75°$.
9. $A = 42°20'$, $B = 57°30'$, $C = 80°10'$. **11.** 4 in., 5 in. **13.** 11 in.
15. 55°30', 59°50', 64°40'.

EXERCISE 18-4. PAGE 304

1. 88.41. **3.** 243,900. **5.** 9285. **7.** 90. **11.** 54 ft. **15.** 45 ft.
17. 38 sq ft, 12 sq ft.

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